Tribhuvan University
Institute of Science and Technology
Model Question

Bio Mathematics | Full Marks: 100
MAT 407 | Pass Marks: 35
Level B.Sc. 4th Year | Time: 3 Hours.

Attempt all questions.

1. (a) The population of elephants in Chitwan National park is 200, find the population after 5 years, if the growth rate is 5% per year. [5]

   (b) In the model $\Delta P = 1.3P(1 - P/10)$, what values of P will cause ΔP to be positive? Negative? Why does this matter biologically? [5]

2. (a) What are the equilibrium points of $P_{t+1} = P_t + 0.7P_t(1 - P_t/10)$ and discuss their stability. [5]

   (b) Solve the logistic differential equation [5]

$$\frac{dN}{dt} = rN[1 - N/K],\ N(0) = N_0$$

3. In a study of insect population, the individuals progress from egg to larva over one step, and larva to adult over another. Finally, adults lay eggs and die in one more time step. Formulate the problem, for which 4% of the eggs survive to become larva , only 39% of the larva make to adulthood, and adults on average produce 73 eggs each. If E_t, L_t and A_t represent the number of eggs, number of larvae and the number of adults respectively, (a) prove that [4+ 4 + 2]

$$A_{t+1} = (0.39)(0.04)(73)A_t = 1.1388A_t$$

   (b) Express the model int he form $x_{t+1} = Px_t$

4. For the matrix

$$P = \begin{pmatrix} 0.9925 & 0.0125 \\ 0.0075 & 0.9875 \end{pmatrix}$$

   find eigenvalues and eigenvectors.

5. Derive the predator-prey model:

$$P_{t+1} = P_t + rP_t(1 - P_t/K) - sP_tQ_t,$$

$$Q_{t+1} = Q_t - uQ_t + vP_tQ_t$$

   where r, s, u, v, K are positive constants and $u < 1$, and all the symbols have their usual meanings. Find their equilibrium points. [6 + 4]

Or

For the predator - prey model

$$P_{t+1} = P_t(1 + 1.3(1 - P_t))0.5P_tQ_t,$$

$$Q_{t+1} = 0.3Q_t + 1.6P_tQ_t$$

(a) compute the Jacobian matrix (b) Evaluate the Jacobian matrix at the equilibrium points. [4+ 6]

6. Consider the 20-base sequence $AGGGATACATGACCCATACA$. [10]

   (a) Use the first five bases to estimate the four probabilities p_A, p_G, p_C, and p_T .

   (b) Repeat part (a) using the first 10 bases.

   (c) Repeat part (a) using all the bases.

7. For n terminal taxa, the number of unrooted bifurcating trees is

$$1 \cdot 3 \cdot 5 \cdots (2n-5) = \frac{(2n-5)!}{2^{n-3}(n-3)!}$$

   Make a table of values and graph this function for $n \leq 10$. [10]

8. Find the probability that the progeny of $DdWw \times ddWw$ is dwarf with round seeds. [10]

Or

Assuming births of each sex are equally likely, a two-child family may have 4 outcomes in the sexes of the children. (a) List the outcomes and give the probability of each. (b) What is the probability that at least one child is a female? (c) What is the probability that the youngest child is a female? (d) What is the conditional probability that the youngest child is a female, given that at least one child is a female? (e) What is the conditional probability that at least one child is a female, given that the youngest child is a female?

9. Derive the SIR model. Find the threshold value. [5+ 5]

Or

Derive SI and SIS models. Solve for all equilibria (S^*, I^*). [7+3]

10. Plot the three points $(1, 1), (0, 3)$, and $(1, 4)$. Then, find the least squares, best-fit line for them. Draw a graph of the line to your plot. [10]

Or

Drug levels in the bloodstream are typically observed to decay exponentially with time from the administration of a dose. A difference equation model that describes this (and gives further reason to try to fit the data of Table

t (day)	0	1	2	3	4
y (mg/l)	200	129	-	58	33

to an exponential curve) is $y_{t+1} = (1-r)y_t$, where r is the percentage of the drug that is absorbed by tissue or broken down by metabolization during one time step. (a) If the initial amount of the drug is y_0, explain why this model leads to $y_t = y_0(1-r)^t$. (b) Letting $k = \ln(1-r)$ and $a = y_0$, show this is equivalent to $y_t = ae^{kt}$. (c) Explain why $0 < r < 1$ for this model, and then why $k < 0$.

MODEL QUESTION
Tribhuvan University

Four year Bachelor Level/ Science & Tech./ Year IV — Full Marks: 100
Mathematical Economics (MAT. 408) — Pass Marks: 35
NEW COURSE — Time: 3 Hrs

Attempt ALL the questions.

1. (a) Define market equilibrium. Write two-commodity market model and extract the equilibrium condition of the model. [1+5]

   (b) Extract the equilibrium solution of the following model. [4]

$$
\begin{aligned}
Q_{d1} &= 10 - 2P_1 - 2P_1 + P_2 \\
Q_{s1} &= -2 + 3P_1 \\
Q_{d2} &= 15 + P_1 - 2P_1 - P_2 \\
Q_{s2} &= -1
\end{aligned}
$$

2. Consider the situation of a mass layoff where 1200 people become unemployment and now begin a job search. In this case there are two states: employed (E) and unemployed (U) with an initial vector $x_0' = [E\ U] = [0 \quad 1200]$. Suppose that in any given period an unemployed person will find a job with probability 0.7. Additionally, people who find themselves employed in any given period may lose their job with a probability of 0.1.

   (a) Set up the Markov transition matrix for this problem.

   (b) What will be the number of unemployment people after (i) 2 periods; (ii) 10 periods?

   (c) What is the steady-state level of unemployment? [5+4+1]

3. (a) Define marginal-cost and average-cost. Given the total-cost function $C = Q^3 - 5Q^2 + 12Q + 75$, write out a variable-cost(VC) function. Find the derivative of the VC function and interpret the economic meaning of derivative.

   (b) Write the economical interpretation of partial derivatives. Use Jacobian determinants to test the existence of functional dependence between the following paired functions.
   $y_1 = 3x_1^2 + x_2; y_2 = 9x_1^4 + 6x_1^2(x_2 + 4) + x_2(x_2 + 8) + 12$ [5+5]

4. What are the main assumptions of the IS-LM national-income model? Derive the slope of IS and LM curves, write the equilibrium identities of the curves. Hence, find the comparative-static derivatives of the model. [1+3+2+4]

OR

Define differentials and point elasticity. Let the equilibrium condition for national income be $S(Y) + T(Y) = I(Y) + G_0 \quad (S', T', I' > 0 : S' + T' > I')$ where $S, Y, T, I,$ and G stand for saving, national income, taxes, investment, and government expenditure, respectively. All derivatives are continuous.

   (a) Interpret the economic meaning of the derivatives S', T', and I'.

   (b) Check whether the conditions of the implicit-function theorem satisfied. If so, write the equilibrium identity.

   (c) Find $\frac{dY^*}{dG_0}$ and discuss its economic implications. [2+5+3]

5. Formulate the wine storage problem and extract the maximization conditions for the problem. [2+8]

OR

Write the first-order and second-order conditions for extremum of more than two variables. Find the extreme values of $Z = -x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$. Check whether they are maxima or minima by the determinantal test. [4+6]

6. A two-product firm faces the following demand and cost functions: $Q_1 = 40 - 2P_1 - P_2$; $Q_2 = 35 - P_1 - P_2$; $C = Q_1^2 + 2Q_2^2 + 10$
   (a) Find the output levels that satisfy the first-order condition for maximum profit.
   (b) Check the second-order sufficient condition. Can you conclude that this problem possesses a unique absolute maximum?
   (c) Find the maximum profit? [4+4+2]

7. (a) Find the extremum of the optimization problem $z = xy$, subject to $x + y = 6$ using the Lagrange-multiplier method.
   (b) Check the optimality of the programming $z = xy$, subject to $x + 2y = 2$ using the bordered Hessian. [5+5]

8. State the Kuhn-Tucker sufficiency conditions for concave programming. Solve the following problem applying the Kuhn-Tucker conditions. [2+8]

$$\begin{aligned} \text{Minimize } C &= (x_1 - 4)^2 + (x_2 - 4)^2 \\ \text{subject to } & \quad 2x_1 + 3x_2 \geq 6 \\ & \quad -3x_1 - 2x_2 \geq -12 \\ & \quad x_1, x_2 \geq 0 \end{aligned}$$

OR

State and establish the Roy's identity. Consider a consumer with the utility function $U = xy$, who faces a budget constraint of B and is given price P_x and P_y. Does the Roy's identity hold for the following choice problem? [5+5]

$$\begin{aligned} \text{Maximize } U &= xy \\ \text{subject to } & \quad P_x x + p_y y = B \end{aligned}$$

9. (a) Find the present value of continuous revenue flow lasting for y years at the constant rate of D dollars per year and discounted at the rate of r per year.
   (b) Find the present value of a perpetual cash flow of:
      i. \$1450 per year, discounted at $r = 5\%$.
      ii. \$2460 per year, discounted at $r = 8\%$. [5+5]

10. Write the basic premises of Domar's growth model and extract the solution to the model. [4+6]

OR

Let the demand and supply functions

$$\begin{aligned} Q_d &= 40 - 2P - 2P' - P'' \\ Q_s &= -5 + 3P \end{aligned}$$

with $P(0) = 12$ and $P'(0) = 1$
   (a) Find the price path, assuming market clearance at every point of time.
   (b) Is the time path convergent? Mention its fluctuation. [7+3]

LECTURE PROGRAM ON

MODERN ALGEBRA (COURSE NO. MAT 401)

FOR

FOUR YEAR B.SC. MATHEMATICS

(IV YEAR)

TRIBHUVAN UNIVERSITY

Prepared by Central Department of Mathematics,

Tribhuvan University, Kirtipur, Kathmandu, Nepal

Tribhuvan University
Institute of Science and Technology
Course of Study for Four Year Mathematics

Course Title: Modern Algebra **Full Marks:** 100
Course No. : MAT 401 **Pass Mark:** 35
Level **:** B.Sc. **Year:** IV
Nature of Course: Theory **Lectures: 150 Hrs.**

Course Description
This course is designed for fourth year of B.Sc. four years level. The main aim of this course is to provide knowledge of modern algebra and Theory of equations.

Course Objective
The main objectives of this course structure is to enable the students to;
- (i) develop in-depth knowledge and good theoretical background in algebra,
- (ii) make interest in and promote enjoyment of algebra and its applications in various branches of mathematics and physical and social sciences,
- (iii) get associated with teaching in the field related to algebra,
- (iv) compare with graduates from various other universities in the field of algebra.

Course Contents

Unit 1 Groups and Subgroups: Introduction and examples, Binary operations, Isomorphic binary structures, Groups, Subgroups, cyclic groups, Generating sets and Cayley diagraphs. [15 Lectures]

Unit 2 Permutations, Cosets, and Direct Product: Groups of permutations, Orbits, Cycles, and the alternating groups, Cosets and Theorem of Lagrange's, Direct products. [15 Lectures]

Unit 3 Homomorphism and Factor Groups: Homomorphisms, Factor groups, Factor group computations and simple groups. [15 Lectures]

Unit 4 Rings and Fields: Rings and fields, Integral domains, Fermat's and Euler's theorems, The field of quotients of an integral domain, Rings of polynomials, Factorization of polynomials over a field. [15 Lectures]

Unit 5 Ideals and Factor Rings: Homomorphisms and factor rings, Prime and maximal ideals. [15 Lectures]

Unit 6 Extension Fields: Introduction to extension fields, Algebraic extensions. [15 Lectures]

Unit 7 Advanced Group Theory: Isomorphism theorem, Sylow theorem (No Proof), Applications of Sylow theory. [15 Lectures]

Unit 8 Factorization: Unique factorization domains, Euclidean domains, Gaussian integers. [15 Lectures]

Unit 9 Theory of Polynomial Equations: Polynomial over an integral domain, division algorithm, division of a polynomial, zero of a polynomial, Rolle's theorem(no proof),properties of equations, Descartes rule of signs, relation between roots and coefficients, application to the solution of an equation, symmetric function of roots, transformation of equations, transformation in general, multiple roots, sum of the power of roots, reciprocal equations, Binomial equation.

[15 Lectures]

Unit 10 Cubic and Biquadratic Equations: Algebraic solution, algebraic solution of the cubic, nature of roots of cubic, equation of square difference of cubic, nature of roots from Cardan's solution and application to the numerical examples, solution by symmetric functions of roots, solution of the biquadratic and the radical. [15 Lectures]

Text Books

1. John. B. Fraleigh; *A First Course in Abstract Algebra*, Seventh Edition, Pearson.
2. R.M. Shrestha & S. Bajracharya; *Linear Algebra, Groups, Rings & Theory of Equations*, Sukunda Pustak Bhavan, Kathmandu.
3. T.P. Nepal, C.R. Bhatta & Ganga Ram D.C.; *A Text Book on Algebra*, Pradhan Book House Exhibition Road, Kathmandu.

Reference Books

4. H.N. Bhattarai & G.P. Dhakal; *Undergraduate Algebra,* Vidharthi Pustak Bhandar, kathmandu.
5. I.N. Herstein; *Topics in Algebra,* Vikas Publication, India.
6. N.S. Gopal Krishan; *University Algebra,* Orient Longman, India.
7. P. B.Bhattacharya, S.K. Jain & S.R. Nagpaul; *Basic Abstract Algebra*, Cambridge, 1995.
8. A.R. Vasishtha; *Modern Algebra*, Krishna Prakashan Mandir,Meerut.

Detailed Course

Unit1. Groups and subgroups:

Introduction and examples, Complex numbers, Euler's formula, Algebra on circles, Exercise1 (1-33, 38, 40), Binary operations: definition and examples, Exercise 2 (1-13, 24. 26), isomorphism binary structures: Exercise 3 (1-18), Groups: definition and examples, Elementary properties of groups,

Theorem 4.15 If G is a group with binary operation *, then the left and right cancellation laws hold in G.

Theorem 4.16 If G is a group with binary operation *, and if a and b are any elements of G, then the linear equations a*x= b and y*a= b have unique solutions x and y in G.

Theorem 4.17 The identity element and inverse of each element are unique in a group.

Corollary 4.18 Let G be a group. For all a, b ∈ G, we have $(a * b)^{-1} = b^{-1} * a^{-1}$.

Finite groups, order of a group, order of an element of a group and group tables, Exercise 4 (1-19, 25, 29, 30), Subgroups: notation and terminology, Subsets and subgroups: examples,

Theorem 5.14 A subset H of a group G is a subgroup of G if and only if

(a) H is closed under the binary operation of G,
(b) The identity element e of G is in H,
(c) For all a ∈ H it is true that a^{-1} ∈ H also.

cyclic subgroups,

Theorem 5.17 Let G be a group and let a ∈ G with $a^n = e$. Then H= $\{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G and every subgroup containing a contains H.

Examples, Exercise 5 (1-12, 14-27, 39, 46, 49, 57), Cyclic groups,

Theorem 6.1 Every cyclic group is abelian.

Theorem 6.3 Division Algorithm for $\mathbb{Z}$, if m is a positive integer and n is any integer, then there exist unique integers q and r such that n= mq+r and $0 \leq r < m$.

Theorem 6.6 A subgroup of a cyclic group is cyclic.

Corollary 6.7 The subgroups of $\mathbb{Z}$ under addition are precisely the groups $n\mathbb{Z}$ under addition for $n \in \mathbb{Z}$.

Greatest common divisor,

Theorem 6.10 Let G be a cyclic group with generator a. If the order of G is infinite, then G is isomorphic to $(\mathbb{Z}, +)$. If G has finite order n, then $G \cong (\mathbb{Z}_n, +_n)$,

Exercise 6 (1-27, 32-37, 48- 51), Generating set and Cayley diagraphs,

Theorem 7.4 The intersection of some subgroups H_i of a group G for $i \in I$ is again a subgroup of G.

Theorem 7.6 If G is a group and $a_i \in G$ for $i \in I$, then the subgroup H of G generated by

$\{a_i \mid i \in I\}$ has element precisely those elements of G that are finite products of integral powers of the a_i, where powers of a fixed a_i may occur several times in the product.

Discussion about Cayley diagraph with examples, Exercise 7 (1-6), Normalizer of a subgroup of a group, index of a subgroup H in a group G, Center of a group with examples

Unit 2. Permutations, Cosets and direct products:

Groups of permutations, Permutation groups,

Theorem 8.5 Let A be a nonempty set, and let S_A be the collection of all permutations of A. Then S_A is a group under permutation multiplication.

Two important examples, Image of H,

Lemma 8.15 Let G and G' be a one-to-one function such that $\varphi(xy)= \varphi(x)\, \varphi(y)$ for all x, y ∈ G. Then $\varphi[G]$ is a subgroup of G' and φ provides an isomorphism of G with $\varphi[G]$.

Theorem 8.16 (Cayley's): Every group is isomorphic to a group of permutations.

Exercise 8 (1-9, 16, 17, 18, 20, 21, 30-36), Orbits, Cyclic, and the alternating groups,

Theorem 9.8 Every permutation σ of a finite set is a product of disjoint cycles.

Even and odd permutation,

Corollary 9.12 Any permutation of a finite set of at least two elements is a product of transposition.

Theorem 9.15 (NO PROOF) No permutation in S_n can be expressed both as a product of even number of transposition and as a product of an odd number of transpositions.

Exercise 9(1-18, 23), Cosets and the theorem of Lagrange, Cosets with examples,

Theorem 10.1 Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by a $\sim_L$ b if and only if $a^{-1}b \in H$. Let $\sim_R$ be defined by a $\sim_R$ if and only if $ab^{-1} \in H$. Then $\sim_L$ and

$\sim_R$ are both equivalence relations on G.

Theorem 10.10 (Theorem of Lagrange) Let H be a subgroup of a finite group G. Then the order of H is a divisor of the order of G.

Corollary 10.11 Every group of prime order is cyclic,

Theorem 10.12 The order of an element of a finite group divides the order of the group.

Index of H in G

Theorem 10.14 Suppose H and K are subgroup of a group G such that $K \leq H \leq G$, and suppose index (H:K) of K in H and index (G:H) of H in G are both finite. Then (G:H) is finite, and (G:K)= (G:H) (H:K).

Exercise 10(1-6, 12, 19, 27, 30-34), Direct product with examples,

Theorem 11.2 Let G_1, G_2, ,G_n be groups. For $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ in $\prod_{i=1}^{n} G_i$, define $(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$ to be the element $(a_1 b_1, a_2 b_2, \ldots, a_n b_n)$. Then $\prod_{i=1}^{n} G_i$ is a group, the direct product of the groups G_i, under this binary operation.

Theorem11.5 (NO PROOF) The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime, that is the gcd of m and n is 1.

Corollary11.6 (NO PROOF) The group $\prod_{i=1}^{n} \mathbb{Z}_m$ is cyclic and isomorphic to $\mathbb{Z}_{m_1 m_2 \ldots m_n}$ if and only if the numbers m_i for i= 1, 2, . . ., n are such that the gcd of any two of them is 1.

Theorem 11.9 Let $(a_1, a_2, \ldots, a_n) \in \prod_{i=1}^{n} G_i$. If a_i is of finite order r_i in G_i, then the order of $(a_1, a_2, \ldots, a_n)$ in $\prod_{i=1}^{n} G_i$ is equal to the least common multiple of all the r_i.

Exercise 11(1-12, 14-20, 32-35),36(a-d), 46.

Unit 3. Homomorphisms and Factor Groups:

Definition of homomorphism, kernel and image of homomorphism, Examples, Properties of homomorphism,

Theorem 13.12: Let $\varphi : G \rightarrow G'$ be a group homomorphism,

(a) If e is the identity element in G, then $\varphi(e)$ is the identity element e' in G',
(b) If $a \in G$, then $\varphi(a^{-1}) = \varphi(a)^{-1}$,
(c) If H is a subgroup of G, then $\varphi[H]$ is a subgroup of G',

(d) If K' is a subgroup of G', then φ^{-1} [K'] isa subgroup of G.

Theorem 13.15 Let φ : G→ G' be a group homomorphism, and let H= Ker(φ). Let a ∈ G. Then the set $\varphi^{-1}[\{\varphi(a)\}] = \{x \in G \mid \varphi(x) = \varphi(a)\}$ is the left coset aH of H, and is also the right coset Ha of H. Consequently, the two partition of G into left cosets and into right cosets of H are the same.

Corollary:13.18 A group homomorphism φ : G→ G' is a one to one map if and only if Ker(φ)={e}.

Normal subgroup with examples

Corollary 13.20 If φ : G→ G' is a group homomorphism, then Ker (φ) is a normal subgroup of G.

Exercise 13(1-29, 32- 42, 44, 50, 51, 52,

Factor groups with examples

Theorem 14.1: Let φ : G→ G' be a group homomorphism with kernel H. Then the cosets of H form a factor group, G/H, where (aH) (bH)= (ab) H. Also, the map μ : G/H→ φ[G] defined by μ (aH)= φ(a) is an isomorphism. Both coset multiplication and μ are well defined, independent of the choices a and b from the cosets.

Theorem14.4: Let H be a subgroup of a group G. Then left coset multiplication is well defined by the equation (aH) (bH)= (ab) H if and only if H is a normal subgroup of G.

Corollary14.5: Let H be a normal subgroup of G. Then the cosets of H form a group G/H under the binary operation (aH) (bH)= (ab) H

Theorem14.9: Let H be a normal subgroup of G. Then γ : G→ G/H given by γ(x)= xH is a homomorphism with kernel H .

Theorem 14.11: (The Fundamental Homomorphism Theorem) Let φ : G→ G' be a group homomorphism with kernel H. Then φ[G] is a group, and μ : G/H→ φ[G] given by μ (gH)= φ(g) is an isomorphism.

If γ : G→ G/H is the homomorphism given by γ (g)= gH, then φ(g)= $\mu\gamma$ (g) for each g ∈ G.

Normal subgroups and inner automorphism

Theorem 14.13 The following are three equivalent conditions for a subgroup H of a group G to be a normal subgroup of G

(a) ghg^{-1} ∈ H for all g ∈ G and h ∈ H
(b) gHg-1= H for all g ∈ G
(c) gH= Hg for all g ∈ G

Exercise 14(No. 1,2,3,4,5,6,7,8 also find quotient groups, 9-15, 23,34,35, 37(a), 38.

Factor group computations and simple groups with examples, definition of simple group and examples

Theorem 15.16 Let φ : G→ G' be a group homomorphism. If N is a normal subgroup G, then φ[N] is a normal subgroup of φ[G]. Also, if N' is a normal subgroup of φ[G], then φ^{-1}[N'] is a normal subgroup of G.

Exercise 15 (1-7, 19)

Unit 4. Rings and Fields:

Rings and fields, definition and basic properties, and examples

Theorem 18.8 If R is a ring with additive identity 0, then for a, b ∈R we have

(a) 0a= a0= 0,
(b) a(-b)= (-a)b= -(ab),
(c) (-a)(-b)= ab

Homomorphism and isomorphism with examples, Commutative ring, unit, division ring, skew field, subring, Exercise 18 (1-21, 23, 24, 25, 33, 38, 39, 40, 48, 49, 50), Divisors of zero, Integral domains, Examples,

Theorem 19.3 In the ring $\mathbb{Z}_n$, the divisors of 0 are precisely those nonzero elements that are not relatively prime to n.

Corollary 19.4 If p is a prime, then $\mathbb{Z}_p$ has no divisors of 0.

Theorem 19.5 The cancellation laws hold in a ring R if and only if R has no divisors of 0.

Theorem 19.9 Every field F is an integral domain.

Theorem 19.11 Every finite integral domain is a field.

Theorem 19.12 If p is a prime, then $\mathbb{Z}_p$ is a field.

The characteristic of a ring with examples

Theorem 19.15 Let R be a ring with unity. If $n.1 \neq 0$ for all $n \in \mathbb{Z}^+$, then R has characteristic 0. If n.1= 0 for some $n \in \mathbb{Z}^+$, then the smallest such integer n is the characteristic of R.

Exercise 19 (1-14, 17), Fermat's and Euler's theorem

Theorem 20.1 (Little Theorem of Fermat) If a $\in \mathbb{Z}$ and p is aprime not dividing a, then p divides ap^{-1}-1, that is, $ap^{-1} \equiv 1 (\text{mod } p)$ for $a \not\equiv 0 (\text{mod } p)$

Corollary 20.2: If a $\in \mathbb{Z}$, then $a^p \equiv a (\text{mod } p)$ for any prime p

Examples

Theorem 20.6 (NO PROOF) The set G_n of nonzero elements of $\mathbb{Z}_n$ that are not 0 divisors form a group under multiplication modulo n.

Theorem 20.8 (Euler's Theorem) If a is an integer relatively prime to n, then $a^{\varphi(n)}$-1 is divisible by n, that is $a^{\varphi(n)} \equiv 1 (\text{mod } p)$.

Theorem 20.10 Let m be a positive integer and let a $\in \mathbb{Z}_m$ be relatively prime to m. For each b $\in \mathbb{Z}_m$, the equation ax= b has a unique solution in $\mathbb{Z}_m$.

Exercise 20 (1-18, 23), The field of quotients of an integral domain,

Lemma 21.2 The relation $\sim$ between elements of the set S as (a, b) $\sim$ (c, d) iff ad=bc is an equivalence relation.

Lemma 21.3 (NO PROOF) For [(a,b)] and [(c,d)] in F, the equations [(a,b)] + [(c,d)]= [(ad+bc,bd)] and [(a,b)] [(c,d)] = [(ac, bd)] give well- defined operations of addition and multiplication on F.

Lemma 21.4 The map i: D$\rightarrow$ F given by i(a)= [(a,1)] is an isomorphism of D with a subring of F.

Exercise 21 (4), Rings of polynomials, Definition

Theorem 22.2 The set R[x] of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication. If R is commutative, then so is R[x], and if R has unity $1 \neq 0$, then 1 is also unity for R[x].

Theorem 22.4 (NO PROOF) (The evaluation Homomorphisms for Field Theory) Let F be a subfield of a field E, let α be any element of E, and let x be an indeterminate. The map φ_α : F[x]$\rightarrow$ E defined by $\varphi_\alpha(a_0+a_1x+\ldots+a_nx^n) = a_0+a_1\alpha+\ldots+a_n\alpha^n$ for $(a_0+a_1x+\ldots+a_nx^n)$ $(a_0+a_1x+\ldots+a_nx^n) \in F[x]$ is a homomorphism of F[x] into E. Also, $\varphi_\alpha(x)= \alpha$, and φ_α maps F isomorphically by the identity map; that is $\varphi_\alpha(a)= a$ for $a \in F$. The homomorphism φ_α is evaluation at α.

Examples, Exercise 22(1-17, 23, 24, 25), Factorization of polynomials over a field,

Theorem 23.1 (Division Algorithm for F[x]) Let $f(x)= a_nx^n+a_{n-1}x^{n-1}+ . . .+a_n$ and $g(x)= b_mx^m+b_{m-1}x^{m-1}+ . . .+b_m$ be two elements of F[x], with a_n and b_m both nonzero elements of F and m > 0. Then there are unique polynomials q(x) and r(x) in F[x] such that f(x)= g(x)q(x)+ r(x), whether either r(x)= 0 or the degree of r(x) is less than the degree m of g(x) .

Example,

Corollary23.3 (Factor Theorem) An element α ∈ F is a zero of f(x) ∈ F[x] if and only if x-a is a factor of f(x) in F[x].

Example

Corollary 23.5(NO PROOF) A nonzero polynomial f(x) ∈ F[x] of degree n can have at most n zeros in a field F.

Irreducible polynomials with examples,

Theorem 23.10 Let f(x) ∈ F[x], and let f(x) be of degree 2 or 3. Then f(x) is reducible over F if and only if it has a zero in F.

Theorem 23.15 (NO PROOF) (Eisenstein Criterion) Let p ∈ ℤ be a prime. Suppose that $f(x)= a_nx^n+. . . .+ a_0$ is in ℤ[x], and $a_n \not\equiv 0 \pmod p$, but $a_i = 0 \pmod p$ for all i < n, with $a_0 \not\equiv 0 \pmod{p^2}$. Then f(x) is irreducible over ℚ.

Examples, Exercise 23 (1- 14, 25, 26-30)

Unit 5. Ideals and Factor Rings:

Homomorphisms and factor rings with examples

Theorem 26.3 Let φ be a homomorphism of a ring R in into a ring R'. If 0 is the additive identity in R, then φ(0)= 0' is the additive identity in R', and if a ∈ R , then φ(-a)= - φ(a). If S is a subring of R, then φ(S) is a subring of R'. Going the other way, if S' is a subring of R', then φ-1[S'] is a subring of R. Finally, if R has unity 1, then φ(1) is unity for φ(R).

Theorem26.5: Let φ: R→ R' be a ring homomorphism, and let H = Ker (φ).Let a ∈ R. Then $\varphi^{-1}[\varphi(a)]$ = a + H= H= a, where a+ H= H+ a is the coset containing a of the commutative additive group <H, +>.

Corollary 26.6: A ring homomorphism φ: R→ R' is a one-to-one map if and only if Ker(φ)= {0}

Theorem 26.7: Let φ: R→ R' be a ring homomorphism with kernel H. Then the additive cosets of H form a ring R/H whose binary operations are defined by choosing representatives. That is, the sum of two cosets is defined by (a+ H)+ (b+ H)= (a+ b)+ H, and the product of the cosets is defined by (a+ H)(b+ H)= (ab+ H). Also, the map : R/H→ φ[R] defined by μ(a+ H)= φ(a) is an isomorphism.

Theorem 26.9: Let H be a subring of the ring R. Multiplication of additive cosets of H is well defined by the equation (a+ H) (b+ H)= ab + H if and only if ah ∈ H and hb ∈ H for all a, b ∈ R and h ∈ H.

Ideal with examples,

Corollary 26.14 Let N be an ideal of a ring R. Then the additive cosets of N form a ring R/N with the binary operations defined by (a+ N)+(b+ N)= (a+ b)+N and (a+ N)(b+ N)= ab+ N.

Factor ring

Theorem 26.16 Let N be an ideal of a ring R. Then γ: R → R/N given by γ(x)= x+ N is a ring homomorphism with kernel N.

Theorem26.17: (Fundamental homomorphism theorem) Let φ : R→ R' be a ring homomorphism with kernel N. Then φ[R] is a ring, and the map μ : R/N→ φ[R] given by μ (x+N)= φ(x) is an isomorphism.

If γ : R→ R/N is the homomorphism given by γ (x)= x+ N, then for each x ∈ R, we have φ(x)= $\mu\gamma$ (x) .

Exercise 26 (1- 4, 9-15, 17-22, 25-29, 37.

Prime and maximal ideals with examples

Theorem 27.5: If R is a ring with unity, and N is an ideal of R containing a unit, then N= R.

Corollary 27.6: A field contains no proper nontrivial ideals.

Theorem 27.9: Let R be a commutative ring with unity. Then M is a maximal ideal of R if and only if R/M is a field.

Corollary 27.11: A commutative ring with unity is a field if and only if it has no proper nontrivial ideals.

Prime ideal with examples,

Theorem 27.15: Let R be a commutative ring with unity, and N ≠ R be an ideal in R. Then R/N is an integral domain if and only if N is a prime ideal in R.

Corollary 27.16: Every maximal ideal in a commutative ring R with unity is a prime ideal.

Theorem 27.17 If R is a ring with unity 1, then the map φ: ℤ → R given by φ(n)= n.1 for n ∈ ℤ is a homomorphism of ℤ into R.

Exercise 27 (1-4, 14(a-g), 15-17.

Unit 6. Extension Fields:

Definition of field, extension field with examples,

Theorem 29.3: (Kronecker's theorem) (Basic goal) Let F be a field and let f(x) be a non constant polynomial in F[x].Then there exists an extension field E of F and an α ∈ E such that f(α) = 0

Algebraic and transcendental elements with examples,

Theorem 29.12: Let E be an extension field of a field F and let α ∈ E. Let φ_α : F[x] → E be the evaluation homomorphism of F[x] into E such that φ_α(a)= a for a ∈ F and φ_α(x)= α. Then α is transcendental over F if and only if φ_α gives an isomorphism of F[x] with a subdomain of E, that is, if and only if φ_α is a one to one map.

The irreducible polynomial for α over F,

Theorem 29.13: Let E be an extension field of a field F and let α ∈ E, where α is an algebraic over F. Then there is an irreducible polynomial p(x) ∈ F[x] such that p(α)= 0. This irreducible polynomial p(x) is uniquely determined up to a constant factor in F and is a polynomial of minimal degree ≥ 1 in F[x] having α as a zero. If f(α)= 0 for f(x) ∈ F[x], with f(x) ≠ 0, then p(x) divides f(x).

The monic polynomial with examples, simple extension with example,

Theorem 29.18: Let E be a simple extension of F(α), and let α be algebraic over F. Let the degree of irr(α, F) be n ≥ 1. Then every element β of E= F(α) can be uniquely expressed in the form $\beta = b_0 + b_1\alpha + b_2\alpha^2 + \ldots + b_{n-1}\alpha^{n-1}$, where the b_i are in F.

Exercise 29 (1-16), 23, 25, 31(a), Algebraic extensions,

Theorem31.3: A finite extension field E of a field F is an algebraic extension of F.

Theorem 31.4: If E is finite extension field of a field F, and K is a finite extension field of E, then K is a finite extension of F, and [K: F] = [K: E][E: F].

Corollary 31.7: If E is an extension field of F, $\alpha \in$ E is algebraic over F, and $\beta \in$ F(α), then deg(β, F) divides deg (α, F).

Theorem 31.11: Let E be an algebraic extension of a field F. Then there exist a finite number of elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ in E such that E= F($\alpha_1, \alpha_2, \ldots, \alpha_n$) if and only if E is a finite dimensional vector space over F, that is, if and only if E is a finite extension of F.

Examples, Exercise 31 (1-13,22, 24, 27)

Unit 7. Advanced group Theory:

Isomorphism theorems,

Theorem 34.2: (First Isomorphism Theorem) Let φ: G→ G' be a homomorphism with kernel K, and let γ_K: G→ G/K be the canonical homomorphism. Then there is a unique isomorphism μ: GK → φ[G] such that φ(x)= $\mu(\gamma_K$(x)) for each x ∈ G.

Lemma 34.3 Let N be a normal subgroup of a group G and let γ: G→ G/N be the canonical homomorphism. Then the map φ from the set of normal subgroups of G containing N to be the set of normal subgroups of G/N given by φ(L)= γ[L] is one to one and onto.

Join of H and N, H ∨ N

Lemma 34.4 If N is a normal subgroup of G, and if H is any subgroup of G, then H ∨ N= HN= NH. Furthermore, if H is also normal in G, then HN is normal in G.

Theorem 34.5: (Second Isomorphism Theorem) Let H be a subgroup of G and let N be a normal subgroup of G. Then (HN)/N≅ H/(H∩ N).

Example

Theorem 34.7: (Third Isomorphism Theorem) Let H and K be normal subgroups of a group G with K ≤ H. Then G/H ≅ (G/K)/ (H/K).

Exercise 34 (1-6), Sylow theorems (NO PROOF), Example, Exercise 36(1-6),

Applications of Sylow theorem,

Theorem 37.6 For a prime number p, every group G of order p^2 is abelian.

Theorem 37.8 If H and K are finite subgroups of a group G, then $|HK|= \frac{(|H|)(|K|)}{|H \cap K|}$

Unit 8. Factorization:

Principal ideal, principal ideal domain generated by a, Examples,

Theorem 27.24: If F is a field, every ideal in F[x] is principal.

Theorem 27.25: An ideal <p(x)> $\neq \{0\}$ of F[x] is maximal if and only if p(x) is irreducible over F.

Theorem 27.27: Let p(x) be an irreducible polynomial in F[x]. If p(x) divides r(x) s(x) for r(x), s(x) ∈ F[x], then either p(x) divides r (x) or p(x) divides s(x). Exercise 27(5-9, 14(f-j), 18, 19)

Definition of principal ideal domain (PID) and unique factorization domain (UFD) with examples,

Theorem 45.17(NO PROOF) Every PID is a UFD.

Definition of primitive polynomial and content with examples,

Lemma 45.23 (NO PROOF) If D is a UFD, then for every non constant f(x) ∈D[x] we have f(x)= (c) g(x), where c ∈D, g(x) ∈ D[x], and g(x) is primitive. The element c is unique up to a unit factor in D and is the content of f(x). Also, g(x) is unique up to a unit factor in D.

Lemma 45.25: (Gauss's Lemma) If D is a UFD, then a product of two primitive polynomials in D[x] is again primitive. Exercise 45 (1-9, 11-17, 21)

Euclidean domains with examples,

Theorem 46.4: Every Euclidean domain is a PID.

Corollary 46.5: A Euclidean domain is a UFD.

Theorem 46.6 For a Euclidean domain with a Euclidean norm ν, ν (1) is minimal among all ν (a) for nonzero a ∈ D, and u ∈ D is a unit if and only if ν (u)= ν (1).

Relation between Euclidean domain, UFD and PID,

Theorem 46.9: (NO PROOF) (Euclidean Algorithm)

Examples, Exercise 46 (1-5, 7,9,11,18)

Gaussian integers with examples

Lemma 47.2 In $\mathbb{Z}[i]$, the following properties of the norm function N hold for all $\alpha,\beta \in \mathbb{Z}[i]$:

(a) $N(\alpha) \geq 0$

(b) $N(\alpha)= 0$ if and only if $\alpha= 0$

(c) $N(\alpha\beta)= N(\alpha)N(\beta)$.

Lemma 47.3: Z[i] is an integral domain.

Exercise 47 (1-4)

Unit 9. Theory of polynomial Equations

9.1 Polynomial over an integral domain, Division algorithm.

Theorem: The set D[x] of all polynomials in x over an integral domain D, is an integral domain.

Theorem: For any two polynomials f(x) and g(x) $\neq 0$ over a field K, there exist unique polynomials q(x) and r(x) such that f(x) = g(x)q(x)+ r(x) where r(x) is zero or of degree less than that of g(x).

Theorem: (Remainder Theorem) The value of a polynomial(x) at x= c is equal to the remainder obtained on dividing f(x) by x- c.

Theorem: (Factor Theorem) x- c is a factor of a polynomial f(x) if and only if f(x) is zero.

9.2 Division of a polynomial, Zero of a polynomial, Rolle's Theorem (no proof),Properties of equations, Properties of equations, Descartes rule of signs.

Theorem: α is a root of the equation f(x) = 0 if and only if the polynomial f(x) is divisible by x- α.

Theorem: A polynomial over ℂ, of degree n, has exactly n zeros in ℂ.

Theorem: In an equation f(x) = 0 with real coefficients, imaginary roots occur in conjugate pairs.

9.3 Related problems discussion.

9.4 Relation between roots and coefficients, Application to the solution of an equation with related problems discussion.

9.5 Symmetric function of roots, Transformation of equations with related problems discussion.

9.6 Transformation in general, multiple roots, Sum of the power of roots with related problems discussion.

9.7 Reciprocal equations, Binomial equation with related problems discussion.

Unit 10. Cubic and Biquadratic Equations

10.1 Algebraic solution, Algebraic solution of the cubic, Nature of roots of cubic.

10.2 Equation of square difference of cubic, Nature of roots from Cardan's solution and application to the numerical examples with related problems discussion.

10.3 Cardan method with related problems discussion.

10.4 Solution by symmetric functions of roots, Solution of the biquadratic and the radical.

10.5 Related problems discussion about Ferrari's method.

10.6 Related problems discussion about radical (Euler's method).

10.7 Related problems discussion about Descarte's method.

Some Important Guide Lines:

All units are equally important. **There will be 10 questions with four OR parts in any questions. All the questions are compulsory.** On the basis of this guideline, we enclose a set of model question for **MAT 401 (Modern Algebra)**.

MODEL QUESTION

Tribhuvan University

Four year Bachelor Level/ Science & Tech./ Year IV Full Marks: 100

Modern Algebra (MAT. 401) Pass Marks: 35

NEW COURSE Time: 3Hrs.

Attempt ALL the questions.

1. Define group with an example. Let S= ℝ- {-1}. Define * on S by a*b= a+ b+ a b. Then
   (a) Show that * gives a binary operation on S,
   (b) Show that (S, *) is a group,
   (c) Find the solution of the equation 2*x*3= 7 in S. [2+ 1+ 4+ 3]

   OR

   Define cyclic group with an example. Prove that every cyclic group is abelian. Also, prove that a subgroup of a cyclic group is cyclic. Verify these result by taking multiplicative group G={1, -1, i, -i} where i^2= -1 1+2+4+3]
2. When do you say two groups G and G' are isomorphic? Prove that every group is isomorphic to a group of permutations. Also, show that $\mathbb{Z}_3$ is isomorphic to A_3. [2+6+2]
3. Define homomorphism of a group. Let φ: $\mathbb{R}^* \rightarrow \mathbb{R}^*$ where $\mathbb{R}^*$= ℝ- {0}is a group under multiplication operation. Show that φ is a group homomorphism. Let H be a subgroup of a group of G. Then prove that left coset multiplication is well defined by the equation (aH) (bH)= (ab)H if and only if H is a normal subgroup of G. [2+3+5]
4. What is an integral domain? Give an example. Prove that every field F is an integral domain. Also, prove that every finite integral domain is a field. [2+3+5]

   OR

   Define the Euler phi- function with an example. If a is an integer relatively prime to n, then prove that $a^{\varphi(n)}$-1 is divisible by n, that is $a^{\varphi(n)} \equiv$1(mod n). Verify this result by taking n= 12 and a= 7. [2+5+3]
5. Define maximal ideal with an example. Let R be a commutative ring with unity. Then prove that M is a maximal ideal of R if and only if R/M is a field. Verify that 2 ℤ is a maximal ideal of a ring ℤ if and only if ℤ/2 ℤ is a field. [2+5+3]
6. Define field and extension field with an example of each. Let F be a field and let f(x) be a non constant polynomial in F[x]. Then prove that there exists an extension field E of F and $\alpha \in$ E such that f(α)= 0. [3+7]

OR

Define algebraic extension of a field F. If E is a finite extension field of a field F, and K is a finite extension field of E, then show that K is a finite extension of F, and [K:F]= [K:E] [E: F]. [2+8]

7. Let H be a subgroup of a group G and let N be a normal subgroup of G. Then prove that (HN)/ N ≅ H/ (H∩N). Verify this result by taking G= $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, H= $\mathbb{Z} \times \mathbb{Z} \times \{0\}$ and N= $\{0\} \times \mathbb{Z} \times \mathbb{Z}$. [8+2]
8. Prove that every Euclidean domain is a principal ideal domain (PID). Is the function ν for $\mathbb{Z}$ given by $\nu(n)= n^2$ for non zero n ∈ $\mathbb{Z}$ an Euclidean norm ? Justify. [6+4]
9. Find the relation between roots and coefficients of the equation $f(x)= p_n+p_{n-1}x+. . . + p_2x^{n-2}+ \quad p_1x^{n-1}+ x^n$. Remove the second term of the equation $x^3+6x^2+12x- 19= 0$ and solve the equation. [2+3+5]

OR

What is the symmetric function of the roots of an equation? If α, β and γ are the roots of the equation $x^3-px^2+qx-r= 0$, find the values of $\sum \alpha^2$ and $\sum \alpha^3$. [2+3+5]

10. Explain Cardan‘s method of solving the cubic equation $ax^3+ 3bx^2+ 3cx+ d= 0$ ($a\neq 0$). Solve by the Cardan’s method the equation $x^3+6x^2+3x+2= 0$. [5+5]

LECTURE PROGRAM

ON

MATHEMATICAL ANALYSIS (COURSE NO: MAT 402)

FOR

FOUR YEAR B.Sc. IN MATEMATICS

(IV YEAR)

TRIBHUVAN UNIVERSITY

Prepared by Central Department of Mathematics

Tribhuvan University, Kathmandu, Nepal

Tribhuvan University
Institute of Science & Technology
Course of Study for Four Year Mathematics

Course Title: Mathematical Analysis **Full Marks:** 100
Course No. : MAT 402 **Pass Mark:** 35
Level : B.Sc. **Year:** IV
Nature of Course: Theory **Period per week:** 9 Lecture Hrs.

Course description
This course is designed for fourth year of Four years B.Sc. program. The main aim of this course is to provide advanced knowledge of analysis to students offering mathematics as a major subject. Prerequisite for this course is Real Analysis, which the students have studied in the third year.

Course objectives
The general objectives of this course is
a) To develop theoretical knowledge and analytical skill in the emerging areas of mathematics
b) To raise interest of students in the field of analytical world so that they can take up any course easily in modern mathematics.
c) To acquire and develop skill in the use and understanding of mathematical language.
d) To acquire knowledge an understanding of the language of mathematical terms, symbols, statements formulae, definitions, logic etc.
e) To construct solutions and proofs with their own independent efforts.
f) To prepare a sound base for higher studies in Mathematics.

Course Contents

Unit 1 Euclidean spaces and metric spaces: **[12 Lecture hours]**
Set R^n, Algebraic structure of R^n, Metric structure of R^n, Cauchy-Schwarz Inequality, Topology in R^n, Metric spaces, Point set topology in metric spaces.

Unit 2 Compactness **[8 Lecture hours]**
Bolzano-Weiererstrass theorem, Cantor intersection theorem, Lindelof covering theorem, Heine-Borel covering theorem, Compactness in R^n, Compactness of a metric space.

Unit 3 Limits and Continuity: **[20 Lecture hours]**
Convergent sequence in a metric space. Cauchy sequences, Complete metric spaces, Sequences and Compactness, Bolzano-Weiererstrass theorem for sequences, Limits of a function, Continuous functions, Continuity and inverse images, Functions continuous on compact sets, Bolzano's theorem and intermediate value theorem, Uniform continuity, Uniform continuity and compact sets.

Unit 4 Multivariable Differentiation: **[16 Lecture hours]**
Linear operator and its matrix representation, Total derivative, Partial Derivatives, Directional derivatives, Jacobean matrix, Mean Value theorem, Higher order partial derivatives.

Unit 5 Functions of Bounded Variation: **[9 Lecture hours]**
Properties of monotonic functions, functions of bounded variation, Total variation, Its additive property. Total variation on $[a, x]$ as a function of x, Functions of bounded variation expressed as the difference of increasing functions, Continuous function of bounded variation.

Unit 6. Riemann-Stieltjes Integration **[22 Lecture hours]**

Riemann-Stieltjes integrals, Linear properties, Integration by parts, Change of variable, Reduction to a Riemann integral, Step-functions as integrators, Increasing integrators, Upper and lower integrals, Riemann's condition, Comparison theorems, Necessary and sufficient conditions for existence of Riemann-Stieltjes integrals, Mean Value theorem, Integral as a function of the interval, Second Fundamental theorem, Second Mean Value theorem.

Unit 7 Sequences and series of functions **[19 Lecture hours]**

Sequences of Functions: Pointwise convergence, Uniform convergence, Criterion for non-uniform convergence, Cauchy Condition for Uniform Convergence, Uniform convergence and continuity, Uniform convergence and integration, Uniform convergence and differentiation.

Series of functions: Uniform convergence of series of functions, Cauchy condition, Weierstrass *M*-test, Dirichlet's test, and Abel's test for uniform convergence. Uniform convergence and continuity, Uniform convergence and integration, Uniform convergence and differentiation.

Unit 8. Improper Integrals **[18 Lecture hours]**

Classification of improper integrals, Convergence, Divergence, Application of Fundamental Theorem of calculus, Simple properties, Conditions and tests for convergence, Absolute convergence, Abel's test and Dirichlet's test.

Unit 9. Complex Numbers and Functions **[10 Lecture hours]**

Algebraic and Geometric properties of complex numbers, Polar coordinates and Eulers formula, Products and quotients in exponential form, Roots of complex numbers, Regions in the complex plane, Complex functions, Complex functions as mappings.

Unit 10. Analytic Functions **[16 Lecture hours]**

Limits and Continuity, Differentiability, Cauchy-Riemann Equations, Sufficient conditions for differentiability, Analytic functions, Reflection principles, Harmonic functions

Text books

1. Tom Apostol, *Mathematical Analysis*, Narosa Publishing House, India.
2. David V. Widder, *Advanced Calculus*, Prentice Hall.
3. James Ward. Brown and Ruel V.Churchill, *Complex Variables and its Applications,* McGraw-Hill.Inc.

Reference books

4. Pahari,N.P., *A Textbook of Mathematical Analysis*, Sukunda Pustak Bhawan, Kathmandu.
5. Brian S. Thomson, Judith B. Bruckner, Andrew M. Bruckner, *Elementary Real Analysis*
6. R. G. Bartle, *The Elements of Real Analysis*, John Wiley and Sons.
7. S. Ponnusamy, *Foundations of Mathematical Analysis*, Springer.
8. V. A. Zorich, *Mathematical Analysis* I and II, Springer.
9. Dennis G.Zill and Patrick D.Shanahan,*Complex Analysis with Applications* , Jones and Bartlett Publisher.
10. John H.Mathews and Russel W.Howell*, Complex Analysis for Mathematics and Engineering*, Jones and Bartlett Learning.

♦ ♦ ♦

DETAILED COURSES

UNIT 1 - POINT SET TOPOLOGY [12 hrs]

1.1 Set $\mathbb{R}^n$. [1.5 hrs]

(a) Definition of n-dimensional point and $\mathbb{R}^n$.

(b) Algebraic structure of $\mathbb{R}^n$**:** Definition of equality, sum, multiplication by scalars, difference, zero, inner product, norm and distance for n-dimensional points.

(c) Vector space and Algebraic properties of $\mathbb{R}^n$.

1. Theorem: (Statement only) $\mathbb{R}^n$ *forms a vector space over the field of real numbers with respect to the coordinatewise addition and scalar multiplication.*

2. Theorem (Statement only): *For any* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and any real number* a,

(i) $\mathbf{x}.\mathbf{y} = \mathbf{y}.\mathbf{x}$ (*Commutative law*)

(ii) $\mathbf{x}.(\mathbf{y} + \mathbf{z}) = \mathbf{x}.\mathbf{y} + \mathbf{x}.\mathbf{z}$ (*Distributive law*)

(iii) $(a\mathbf{x}).\mathbf{y} = a(\mathbf{x}.\mathbf{y}) = \mathbf{x}.(a\,\mathbf{y})$

(iv) $\mathbf{x}.\mathbf{x} = 0$ for $\mathbf{x} = \mathbf{0}$ and $\mathbf{x}.\mathbf{x} \neq 0$ for all $\mathbf{x} \neq \mathbf{0}$.

3. Theorem: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and a be a real number. Then

i. $\| \mathbf{x} \| \geq 0$ and $\| \mathbf{x} \| = 0$ iff $\mathbf{x} = \mathbf{0}$.

ii. $\| \mathbf{x} \| = \| -\mathbf{x} \|$

iii. $\| a\mathbf{x} \| = | a | \, \| \mathbf{x} \|$ (*Absolute Homogenity*)

iv. $\| \mathbf{x} - \mathbf{y} \| = \| \mathbf{y} - \mathbf{x} \|$. (*Symmetry*)

v. $\| \mathbf{x} + \mathbf{y} \|^2 + \| \mathbf{x} - \mathbf{y} \|^2 = 2 (\| \mathbf{x} \|^2 + \| \mathbf{y} \|^2)$. (*Parallelogram Equality*)

vi. If $\mathbf{x}.\mathbf{y} = 0$, then $\| \mathbf{x} + \mathbf{y} \|^2 = \| \mathbf{x} \|^2 + \| \mathbf{y} \|^2$ (*Pythagorean identity*)

4. Theorem: Let $\mathbf{x}$ and $\mathbf{y}$ denote two points in $\mathbb{R}^n$. Then,

i. $| \mathbf{x} . \mathbf{y} | \leq \| \mathbf{x} \| \, \| \mathbf{y} \|$. (*Cauchy-Schwarz's Inequality*)

ii. $\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|$. (*Triangle Inequality*)

iii. $| \, \| \mathbf{x} \| - \| \mathbf{y} \| \, | \leq \| \mathbf{x} - \mathbf{y} \|$.

(d) Definition of unit co-ordinate vectors in $\mathbb{R}^n$.

1.2 Open Balls and Open Sets in $\mathbb{R}^n$. [2.5 hrs]

(a) Definition of open n-balls, closed balls and spheres with examples and geometrical shape for different n.

(b) Definitions of (i) interior point & interior (ii) exterior point & exterior (iii) boundary point & boundary of a set in $\mathbb{R}^n$.

(c) Definition of open and closed sets in $\mathbb{R}^n$ with examples

Theorem: *Let* $S \subseteq \mathbb{R}^n$, then

(i) *S is open* in $\mathbb{R}^n$ *if, and only if, S = int S.*

(ii) *The interior of S in an open subset of S.*

(d) Construction of open and closed Sets in $\mathbb{R}^n$

1. **Theorem:** *The union of arbitrary collection of open sets in* $\mathbb{R}^n$ *is open and the intersection of a finite collection of open sets is open in* $\mathbb{R}^n$.

2. **Theorem:** *The union of a finite collection of closed sets in* $\mathbb{R}^n$ *is closed and the intersection of arbitrary collection of closed sets in* $\mathbb{R}^n$ *is closed in* $\mathbb{R}^n$.

3. Examples to show arbitrary intersection of open sets may not be open and the arbitrary union of closed sets may not be closed.

4. **Theorem:** *If A is closed and B is open, then A − B is closed and B − A is open.*

1.3 Closed Sets and Adherent Points. [3 hrs]

(a) Definition of adherent point and closure; accumulation point and derived set; and isolated point of a set with examples.

1. **Theorem**: *Let* $S \subseteq \mathbb{R}^n$ *and* $\mathbf{x} \in \mathbb{R}^n$, *then* x *is an accumulation point of S if, and only if,* x *is an adherent point of* $S - \{\mathbf{x}\}$.
2. **Theorem:** **x** *is an accumulation point of S if , and only if, every open n-ball B(*x*) contains infinitely many points of S.*
3. Example of an infinite set *with no accumulation points.*
4. **Theorem:** *A set S in* $\mathbb{R}^n$ *is closed if, and only if, it contains all its adherent points.*

(b) Definition of Closure and Derived set of a set with examples.

1. Proof of the relation $\overline{S} = S \cup S'$.
2. **Theorem:** *A set S in* $\mathbb{R}^n$ *is closed if, and only if, S contains all its accumulation points.*
3. **Theorem:** *A set S in* $\mathbb{R}^n$ *is closed if and only if it contains all its boundary points.*
4. **Theorem:** *The derived set S ' of a set S in* $\mathbb{R}^n$ *is a closed set.*

Definition of dense set in $\mathbb{R}^n$ and examples

1.5 Metric Spaces. [1 hr]

(a) Definition of metric space with examples

(b) Examples of metric spaces (Discrete metric space ,$\mathbb{R}^n$ with different types of matrices)

(c) Definition of metric subspace with examples.

1.6 Point Set Topology in Metric-Spaces. [2 hrs]

(a) Definition of open ball in metric spaces and subspaces

(b) Notation $B_S(a; r) = B_M(a; r) \cap S$.

(c) Definition of interior point, open sets and closed sets in metric space with examples.

(d) Construction of open and closed sets in Metric Space.

(e) Definition of adherent point,accumulation point, closure and derived set in a metric space.

1. **Theorem:** *Let (S, d) be a metric subspace of the metric space (M, d) and X, a subset of S. Then X is open in S if, and only if,* $X = A \cap S$ *for some set A which is open in M.*
2. **Theorem:** *Let (S, d) be a metric subspace of a metric space (M, d) and let Y be a subset of S. Then Y is closed in S if, and only if,* $Y = B \cap S$ *for some closed set B in M.*
3. **Theorem:** *The union of any collection of open sets is open and the intersection of a finite collection of open sets is open.*
4. **Theorem:** *The union of a finite collection of closed sets in a metric space is closed and the intersection of any collection of closed sets is also closed.*
5. **Theorem:** *If A is open in metric space M and B is closed in M, then A − B is open in M and B − A is closed in M.*
6. **Theorem:** *For any subset S of a metric space M, the following statements are equivalent:*
   (i) *S is closed in M.*
   (ii) *S contains all its adherent points.*
   (iii) *S contains all its accumulation points.*
   (iv) $S = \overline{S}$.

Exercises. **[2 hrs]**

At least the following problems from Exercise on page 65, Chapter-3 of the book *Mathematical Analysis*, Tom M. Apostol II Edition (1987) should be discussed: 3.9, 3.11, 3.12, 3.26, 3.27, 3.28, 3.29, 3.31

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UNIT 2 - COMPACTNESS [8 HRS]

2.1 Bolzano-Weierstrass Theorem **[1 hr]**

(a) Definition of a bounded set in $\mathbb{R}$, $\mathbb{R}^n$ and Metric spaces

(b) Bolzano-Weierstrass Theorem for Bounded Set in $\mathbb{R}^n$, $n > 1$. In the proof, emphasis should be given for $\mathbb{R}^n$, $n > 1$, by recalling the proof in $\mathbb{R}^1$ which is already taught in MAT 302, third years course.

2.2 The Cantor-Intersection Theorem (Statement and Proof). **[1 hr]**

2.3 Lindelöf Covering Theorem. **[1.5 hrs]**

(a) Definition of covering, open covering, countable covering, finite covering and subcovering with examples.

(b)Theorem: *Let $G = \{A_1, A_2, \ldots\}$ denote a countable collection of all n-balls having rational radii and centres at points with rational co-ordinates. Assume $\mathbf{x} \in \mathbb{R}^n$ and let S be an open set in $\mathbb{R}^n$ such that $\mathbf{x} \in S$. Then, there exists at least one of the n-balls A_k in G contains $\mathbf{x}$ and is contained in S. That is, we have $\mathbf{x} \in A_k \subseteq S$ for some A_k in G.*

(c)Theorem: (Lindelöf covering Theorem in $\mathbb{R}^n$**) (Statement and Proof)**

2.4 Heine Borel Covering Theorem. **[1 hr]**

Theorem: *Let F be an open covering of a closed and bounded set A in $\mathbb{R}^n$. Then there is a finite sub collection of F which also covers A.*

2.5 Compactness in $\mathbb{R}^n$. **[1 hr]**

(a) Definition of compact sets in $\mathbb{R}^n$ with examples.

(b) Theorem: *Let S be a subset of $\mathbb{R}^n$. Then the following three statements are equivalent.*

(i) *S is compact.*

(ii) *S is closed and bounded.*

(iii) *Every infinite subset of S has an accumulation point in* S.

2.6 Compact Subsets of a Metric Space. **[1.5 hrs]**

(a) Definition of open covering, compact set and bounded set in a metric space.

(b) Theorem: *Let S be a compact subset of a metric space M. Then*

(i) *S is closed and bounded.*

(ii) *Every infinite subset of S has an accumulation point in S.*

(c) Example to show that there exists a closed and bounded set in a metric space which is not compact.

(d) Theorem**:** Let X be a closed subset of a compact metric space *M*. Then *X* is compact.

(e)Theorem: *Assume that $S \subseteq T \subseteq M$, (M, d) is a metric space. Then S is compact in (M, d) if, and only if, S is compact in the metric space* (*T, d*).

2.7 Exercises. **[1 hr]**

At least the following problems from the Exercise on page 66 and 68, Chapter-3 of the book *Mathematical Analysis*, Tom M Apostol II Edition (1987) should be discussed: 3.18, 3.19 , 3.38, 3.39, 3.40, 3.41, 3.42.

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UNIT 3 - LIMITS AND CONTINUITY [20 HRS]

3.1 Convergent sequences in metric spaces. [3 hrs]

(a) Definition of convergent sequence with examples, definition of increasing, decreasing sequences, subsequences and bounded sequence.

(b) Sequence $\{1/n\}$ converges in $\mathbb{R}$ but not in (0, 1].

(c) Theorem: (Uniqueness of limit) *A sequence* $\{x_n\}$ *of points in a metric space* (S, d) *can converge to at most one point in S.*

(d) Theorem: *An increasing sequence* $\{x_n\}$ *which is bounded above converges to the supremum of its range* $\{x_1, x_2, \ldots\}$ *and a decreasing sequence* $\{x_n\}$ *which is bounded below converges to the infimum of its range* $\{x_1, x_2, \ldots\}$.

(e) Theorem: *Let* $\{x_n\}$ *be a sequence of points in a metric space (S, d) and let* $x_n \to p$. *If* $T = \{x_1, x_2, \ldots\}$ *is the range of* $\{x_n\}$, *then*

(i) *T is bounded.* (ii) *p is an adherent point of T.*

(f) Theorem: *Let (S, d) be a metric space and* $T \subseteq S$. *Let p be a point in S which is an adherent point of T. Then there is a sequence* $\{x_n\}$ *of points in T converging to p.*

(g) Theorem: *A sequence* $\{x_n\}$ *of points in a metric space* (*S*, *d*) *converges to p if, and only if, its every subsequence* $\{x_{k(n)}\}$ *also converges to p.*

(h) Theorem: (Sequential criterion of closed Set)

Let (S, d) be a metric space and A be a non empty subset of S. Then $p \in \overline{A}$ *if and only if there exists a sequence* $\{x_n\}$ *in A such that* $x_n \to p$.

3.2 Cauchy Sequences. [1.5 hr]

(a) Definition of Cauchy sequence and examples

(b) Theorem (Cauchy's general principle of convergence or Cauchy convergence criterion)

(c) Theorem : *Every convergent sequence in a metric space is Cauchy but not conversely*.

(d) Theorem**:** *In Euclidean space* $\mathbb{R}^k$, *every Cauchy sequence is convergent.*

3.3 Complete Metric Space [1 hr]

(a) Definition of complete metric space with examples.

(b) Theorem: *In a metric space (S, d), every compact subset T is complete.*

3.4 Sequences and Compactness [1 hr]

(a) Definition of sequentially compact metric space with example.

(b) Theorem: *If A metric space (S, d) is sequentially compact, then every infinite subset of S has a limit point.*

(c) Theorem: *Every compact metric space (S, d) is sequentially compact*.

3.5 Bolzano Weierstrass Theorem for Sequence [0.5 hr]

Theorem (Bolzano Weierstrass theorem for sequence): *Every bounded sequence in* $\mathbb{R}$ *has a convergence subsequence.*

3.6 Limit of a Function. [1.5 hr]

(a) Definition of a limit of function in terms of ε, δ and in terms of open balls.

(b) Theorem (Sequential criterion for limit of a function):

Let *(S,* d_S*) and (T,* d_T*)* be two metric spaces, $A \subset S$ and $f : A \to T$ *functions***.** *Let p be an accumulation point of A and assume that* $b \in T$. *Then* $\lim_{x\to p} f(x) = b$ *if, and only if,* $\lim_{n\to\infty} f(x_n) = b$, *for every sequence* $\{x_n\}$ *of points of* $A - \{p\}$ *which converges to p.*

(c) Theorem (Algebra of limit of functions):

Let (*S*, *d*) *be a metric space,* $A \subset S$ *and p be an accumulation point of A.*

Let $f : A \to \mathbb{R}$ *and* $g : A \to \mathbb{R}$ *be two real valued functions and assume that*

$\lim_{x \to p} f(x) = a$ *and* $\lim_{x \to p} g(x) = b$. *Then*

i) $\lim_{x \to p} [f(x) \pm g(x)] = a \pm b;$ ii) $\lim_{x \to p} \lambda f(x) = \lambda a$;

iii) $\lim_{x \to p} [f(x) . g(x)] = ab;$ iv) $\lim_{x \to p} \dfrac{f(x)}{g(x)} = \dfrac{a}{b}$ *if* $b \neq 0$*;and*

iv) $\lim_{x \to p} |f(x)| = |a|$.

3.7 Continuous Functions. [2 hr]

(a) Definition of continuous function in terms of ε, δ and in terms of open balls.

(b) Theorem: *Every function is continuous at an isolated point.*

(c) Theorem (Continuity of Composite Function):

Let (S, d_S), (T, d_T) *and* (U, d_U) *be three metric spaces. Let* $f : S \to T$ *and* $g : f(S) \to U$ *be two functions. If* f *is continuous at a point* $p \in S$ *and* g *is continuous at* $f(p)$, *then the composite function* $h = g o f$ *defined on* S *by the equation* $h(x) = g(f(x))$ *for all* $x \in S$ *is continuous at* p.

(d) Theorem (Sequential criterion for continuity):

Let (S, d_S) *and* (T, d_T) *be two metric spaces,* $f : S \to T$ *be a function and assume that* $p \in S$. *Then* f *is continuous at* p *if, and only if, for every sequence* $\{x_n\}$ *of points of* S *converging to* p, *the sequence* $\{f(x_n)\}$ *of points of* T *converges to* $f(p)$.

3.8 Continuity and Inverse Images of Open or Closed Sets. [2 hr]

(a) Definition of inverse image (or pre-image) of a set and examples.

(b) Theorem (Continuity and inverse image of open (or closed) set:

Let $f : S \to T$ *be a function from a metric space* (S, d_S) *to another metric space* (T, d_T). *Then* f *is continuous on* S *if, and only if, for every open (or closed) set* Y *in* T, *the inverse image* $f^{-1}(Y)$ *is open (or closed) in* S.

(c) Examples to show image of open (or closed) set under a continuous map need not be open (or closed).

3.9 Continuous Functions on Compact Sets. [2 hrs]

(a) Theorem: *The image of a compact set under a continuous function is compact*.

(b) Definition of a bounded vector valued function $\mathbf{f} : S \to \mathbb{R}^k$

(c) Theorem: *Le*t $\mathbf{f} : S \to \mathbb{R}^k$ *be a function from a metric space* S *to a Euclidean space* $\mathbb{R}^k$. *If* $\mathbf{f}$ *is continuous on a compact subset* X *of* S, *then* $\mathbf{f}$ *is bounded on* X.

(d) Theorem: *Let* $f : S \to \mathbb{R}$ *defined on a metric space* (S, d_S) *to Euclidean space* $\mathbb{R}$. *If* f *is continuous on a compact subset* X *of* S, *then* $\exists$ *points* p *and* $q \in X$ *such that*

$f(p) = \text{Infimum } f(X)$ *and* $f(q) = \text{Supremum } f(X)$.

(e) Corollary: *A continuous function attains its maximum and minimum on a compact set.*

(f) Theorem: *Let $f : S \to T$ be a function from a metric space (S, d_S) to another metric space (T, d_T). Assume that f is one to one on S, so that the inverse function f^{-1} exists. If S is compact and if f is continuous on S; then f^{-1} is continuous on $f(S)$.*

(g) Example to show that compactness is essential in above theorem (d).

3.10 Topological Mappings (Homeomorphisms). **[0.5 hr]**

(a) Definition of topological mappings.

(b) Discussion about topological properties and isometry.

3.11 Bolzano's Theorem. **[1.5 hrs]**

(a) Theorem (Sign preserving property for continuous function):

Let f be defined on an interval S in $\mathbb{R}$. Assume that f is continuous at a point c in S and let $f(c) \neq 0$. Then, there exists an one ball $B(c; \delta)$ such that $f(x)$ has the same sign as $f(c)$ in $B(c; \delta) \cap S$.

(b) Theorem (Bolzano's theorem or location of roots theorem):

Let f be a real-valued and continuous function on a compact interval $[a, b]$ in $\mathbb{R}$, and suppose that $f(a)$ and $f(b)$ have opposite signs i.e. $f(a) . f(b) < 0$. Then there exists at least one point $c \in (a, b)$ such that $f(c) = 0$.

(c) Theorem (Intermediate value theorem):

Let f be a real valued continuous function on a compact interval S in $\mathbb{R}$. Suppose that there are two points $a < b$ in S such that $f(a) \neq f(b)$. Then f takes every values between $f(a)$ and $f(b)$ in the interval (a, b).

3.12 Uniform Continuity. **[0.5 hr]**

(a) Definition of uniform continuity and examples

(b) Non-uniform continuity condition

(c) **Theorem** (Uniform Continuity implies continuity):

313 Uniform Continuity and Compact Set. **[1 hr]**

Theorem (Heine theorem for uniform continuity on compact set):

Let $f : S \to T$ be a function from one metric space (S, d_S) to another metric space (T, d_T). Let A be a compact subset of S and assume that f is continuous on A. Then f is uniformly continuous on A.

3.14 Exercises. **[2 hr]**

At least the following problems from Exercise on page 96, Chapter-4 of the book *Mathematical Analysis*, Tom M Apostol II Edition (1987) should be discussed: 4.2, 4.5,4.6, 4.7, 4.9, 4.29, 4.30, 4.50, 4.51, 4.52.

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UNIT 4 - MULTIVARIABLE DIFFERENTIAION [16 HRS]

4.0 Review. **[0.5 hr]**

(a) Review of derivative of vector valued function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ and its limitations.

(b) Review of partial derivatives of function $f : \mathbb{R}^n \to \mathbb{R}$ and its limitations.

4.1 Matrix Representation of a Linear Function **[1 hr]**

(a) Theorem: *Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Then $\mathbf{T}$ can be represented by a $m \times n$*

matrix $m(\mathbf{T})$ *of scalars.*

(b) Derivation of the matrix of the composite map **:** Proof of $m(\mathbf{S}o\mathbf{T}) = m(\mathbf{S})\, m(\mathbf{T})$.

4.1 Directional Derivatives. [0.5 hr]

(a) Definition of directional derivative for functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ in the direction given by **u**.

(b) Consequence of definition (a).

4.2 Directional Derivative and Continuity. [0.5 hr]

(a) Example to show the existence of partial derivatives does not imply the existence of directional derivatives $\mathbf{f}'(c;\mathbf{u})$ in all the directions given by **u.**

(b) Example to show the existence of $\mathbf{f}'(\mathbf{c};\mathbf{u})$ in all **u** does not imply the continuity of **f** at **c.**

4.3 Total Derivative. [2 hrs]

(a) Definition of total derivative with review of error function in one dimensional case.

(b) Theorem: (Equality of Total derivative and Directional Derivative)

If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable at* **c** *in* $S \subseteq \mathbb{R}^n$ *with total derivative* $T_{\mathbf{c}}$, *then the directional derivative* $\mathbf{f}'(\mathbf{c};\mathbf{u})$ *exists for every u in* $\mathbb{R}^n$ *and we have* $\mathbf{f}'(\mathbf{c};\mathbf{u}) = T_{\mathbf{c}}(\mathbf{u})$.

(c) Theorem: *Total derivative of a differentiable function* **f:** $\mathbb{R}^n \to \mathbb{R}^m$ *if exists, is unique.*

(d) Theorem (Differentiability implies continuity):

Let $S \subseteq \mathbb{R}^n$ and *if* $\mathbf{f} : S \to \mathbb{R}^m$ *is differentiable at* c, *then* **f** *is continuous at* c.

4.4 Total Derivatives Expressed in terms of Partial Derivatives. [1 hr]

(a) Theorem: *A function which is differentiable at point* **c** *admits first order partial derivative at that point.*

(b) Prove the relation: $\mathbf{f}'(\mathbf{c})(\mathbf{v}) = \sum_{k=1}^{n} v_k D_k \mathbf{f}(\mathbf{c})$.

4.6 The Jacobian Matrix. [1.5 hr]

(a) Derivation of Jacobian matrix (Matrix in connection with total derivative)

Statement: *Let* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ *be a function differentiable at each point* $\mathbf{c} \in \mathbb{R}^n$ *with total derivative* $\mathbf{T} = \mathbf{f}'(\mathbf{c})$. *Then* **T** *can be represented by an* $m \times n$ *matrix with entries of the components* $\mathbf{Df}(\mathbf{c})$.

(b) Derivation of the inequality $\| \mathbf{f}'(\mathbf{c})(\mathbf{v}) \| \le M \| \mathbf{v} \|$, where $M = \sum \| \nabla f_k(\mathbf{c}) \|$.

4.7 The Chain Rule and its Matrix Form. [1.5 hr]

(a) Theorem: Chain rule for total derivative of **h** = **fog** in terms of the composition of the total derivatives of **f** and **g.**

(b) Matrix form of Chain rule (Expression of the partial derivatives of the components of **h** = **fog** in terms of the partial derivatives of the components of **f** and **g**)

(c) Application of Chain rule: (Statement only)

4.8 The Mean Value Theorem (MVT) for Differentiable Functions. [1.5 hr]

(a) Review of one dimensional MVT and definition of line segment joining **x** and **y** in $\mathbb{R}^n$.

(b) MVT on Multivariable Calculus for Differentiable Functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$

(c) Application of MVT: Statements only (In the proof, the notion of connectedness is used, which is not introduced in our course).

4.9 Sufficient Condition for Differentiability. [1 hr]

(a) Theorem: (Sufficient condition for differentiability):

Let S be the subset of $\mathbb{R}^n$ *and assume* $\mathbf{f} : S \to \mathbb{R}^n$ *be a function. Assume that one of the partial derivatives* $D_1\mathbf{f}, \ldots, D_n\mathbf{f}$ *exists at* $\mathbf{c} \in S$ *and that the remaining* $n-1$ *partial derivatives exist in some n-ball* $B(\mathbf{c})$ *and are continuous at* $\mathbf{c}$. *Then* $\mathbf{f}$ *is differentiable at* $\mathbf{c}$.

4.10 Sufficient Conditions for Equality of Mixed Partial Derivatives. [2 hrs]

(a) Notation $D_{r,k}\,\mathbf{f} = D_r\,(D_k\,(\mathbf{f})) = \dfrac{\partial^2 \mathbf{f}}{\partial x_r \partial x_k}$.

(b) Example to show that mixed partial derivatives $D_{1,2}\,f(x, y)$ and $D_{2,1}\,f(x, y)$ are not necessarily equal, i.e. $f_{xy} \neq f_{yx}$.

(c) Theorem (A sufficient condition for equality mixed partial derivatives):

Young's Theorem: *Assume that* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$. *If both partial derivatives* $D_r\mathbf{f}$ *and* $D_k\mathbf{f}$ *exist in an n-ball* $B(\mathbf{c}; \delta)$ *and if both are differentiable at* $\mathbf{c}$, *then* $D_{r,k}\mathbf{f}(\mathbf{c}) = D_{k,r}\mathbf{f}(\mathbf{c})$.

(d) Theorem (Schwarz's Theorem)**:**

If both partial derivatives $D_r\,\mathbf{f}$ *and* $D_k\,\mathbf{f}$ *exist in an open n-ball* $B(\mathbf{c})$ *and if both* $D_{r,k}\,\mathbf{f}$ *and* $D_{k,r}\,\mathbf{f}$ *are continuous at* $\mathbf{c}$. *Then,* $D_{r,k}\,\mathbf{f} = D_{k,r}\,\mathbf{f}$.

4.11 Taylor's Formula for Functions from $\mathbb{R}^n$ to $\mathbb{R}'$. [1 hr]

(a) Review of Taylor's formula in one dimensional case.

(b) Notation $f''(\mathbf{x}; \mathbf{t}) = \sum_{i=1}^{n}\sum_{j=1}^{n} D_{i,j}\,f(\mathbf{x})t_j t_i$ and similar for higher order.

(c) Proof of Taylor's for formula higher order directional derivatives from $\mathbb{R}^n$ to $\mathbb{R}^1$.

Exercises. [1 hr]

At least the following problems from Exercise on page 362, Chapter-12 of the book ***Mathematical Analysis***, Tom M. Apostol II Edition (1987) should be discussed:
12.3, 12.4, 12.7,12.8, 12.12, 12.24

Note: For this chapter, 1 lecturer hour should be taken from hours alloted for review.

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UNIT 5 FUNCTIONS OF BOUNDED VARIATION:[9 LECTURE HOURS]

5.1 Review of Monotonic Functions. [0.5 hr]

(a) Definition of increasing (non-decreasing) and decreasing (non-increasing) functions.

(b) Definition of strictly increasing and strictly decreasing functions.

(c) Definition of monotonic functions.

(d) Examples related to above definitions.

5.2 Properties of Monotonic Functions. [1 hr]

(a) Theorem: *If* f *is an increasing function defined on* $[a, b]$ *and let* $x_0, x_1, \ldots, x_n$ *be* $n + 1$ *points such that* $a = x_0 < x_1 < \ldots\ldots < x_n = b$, *then*

$$\sum_{k=1}^{n-1}[f(x_k+)-f(x_k-)]\le f(b)-f(a).$$

(b) Theorem: *If f is monotonic on $[a, b]$, then the set of discontinuities of f is countable.*

5.3 Function of Bounded Variation (BV). [2 hrs]

(a) Definition of a partition of a compact interval.

(b) Definition of a function of BV on $[a, b]$.

(c) Theorem: *If f is monotonic on $[a, b]$, then f is of BV on $[a, b]$.*

(d) Theorem: *If f is continuous on $[a,b]$ and if its derivative $f'(x)$ exists and is bounded in (a, b), then f is of BV on $[a, b]$.*

(e)Theorem: *If f is of BV on $[a, b]$, say $\sum|\Delta f_k| \le M$ for all partitions of $[a, b]$, then f is bounded on $[a, b]$. In fact, $|f(x)| \le |f(a)| + M \ \forall\ x \in [a, b]$.*

(f) Examples related to the functions of BV

i. *The function* $f(x) = \begin{cases} x^2 \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$ *is of BV on* $[0, 1]$.

ii. *The function $f(x) = x \cos(\pi/2x)$ if $x \ne 0$ and $f(0) = 0$ is continuous but not of BV on* $[0, 1]$.

iii. *The function $f(x) = x^2 \cos(1/x)$ if $x \ne 0$ and $f(0) = 0$ is continuous and is of BV on* $[0, 1]$.

iv. *Boundedness of $f'(x)$ is not necessary for $f(x) = x^{1/3}$ to be of BV on* $[0, 1]$.

5.4 Total Variation. [1 hr]

(a) Definition of total variation and its consequences.

(b) Theorem: *Let f and g are each function of BV on $[a, b]$. Then so are their sum, difference and product.*

(c) Theorem: *The quotient f/g need not be a function of BV.*

(d) Theorem: *If f is of BV on $[a, b]$ and assume that f is of bounded away from zero. Then $g = 1/f$ is also of BV on $[a, b]$.*

5.5 Additive Property of Total Variation: [1 hr]

(a) Theorem: *Let f be of BV on $[a, b]$ and assume that $c \in (a, b)$. Then f is of BV on $[a, c]$ and on $[c, b]$. And we have $V_f(a, b) = V_f(a, c) + V_f(c, b)$.*

(b) Theorem: *If f is a function of BV on $[a, b]$, then it is so on every closed subinterval $[c, d]$ of $[a, b]$.*

5.6 Total Variation on [*a*, *x*] as a function of *x*: [0.5 hr]

(a) Theorem: *Let f be a function of BV on $[a, b]$. Let V be defined on $[a, b]$ as follows: $V(x) = V_f(a, x)$ for $a < x \le b$ and $V(a) = 0$. Then both V and $V - f$ are increasing function on $[a, b]$.*

5.7 Function of BV expressed as the difference of two increasing functions. [1 hr]

(a) Theorem: *f is of BV on $[a, b]$ if, and only if, f can be expressed as the difference of two increasing functions.*

(b) Theorem: *f is of BV on $[a, b]$ if, and only if, f can be expressed as the difference of two strictly increasing functions.*

(c) Remarks: Representation of a function of BV as a difference of two increasing functions (or strictly increasing functions) is not unique.

5.8 Continuous Function of Bounded Variations. [1 hr]

(a) Theorem: *Let f be of BV on* $[a, b]$. *Define a function V on* $[a, b]$ *by*

$V(x) = V_f(a, x)$ for $x \in (a, b]$ and $V(a) = 0$. *Then every point of continuity of f is also a point of continuity of V and conversely.*

(b) Theorem: *Let f be a continuous function on* $[a, b]$. *Then f is of BV on* $[a, b]$ *iff f can be expressed as the difference of two increasing continuous functions.*

5.9 Exercises. [1 hr]

At least the following problems from Exercise on page 137, Chapter-6 of the book *Mathematical Analysis*, Tom Apostol II Edition (1987) should be discussed: 6.1, 6.3.

Note: If time permits, one can give a list of simple problems for exercise to the students from other books like *'Introduction to Mathematical Analysis'* by Amritava Gupta, *'The Elements of Real Analysis'* by Robert G. Bartle etc.

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UNIT 6 - RIEMANN STIELTJES (R-S) INTEGRATION [22 HRS]

6.1 Review of Riemann Integrals. [0.5 hr]

(a) Definition of partition, Norm of a partition, Refinement of a partition, Lower, Upper and Riemann Riemann sums of a bounded function.

(b) Definition of upper and lower integrals, Riemann integral.

(c) Theorem: (No Proof) Necessary and sufficient conditions for the existence of Riemann integrability.

6.2 Riemann-Stieltjes integrals. [0.5 hr]

(a) Definition of R-S sum of a bounded function f w.r.t. α on $[a, b]$.

(b) Definition of R-S integral of a bounded function f w.r.t. α on $[a, b]$.

6.3 Linear Properties of R-S Integrals. [1.5 hr]

(a) Theorem (Linearity on integrand)

If $f \in R(\alpha)$, $g \in R(\alpha)$ *on* $[a, b]$, *then* $c_1 f + c_2 g \in R(\alpha)$ *on* $[a, b]$, *where* c_1 *and* c_2 *are two constants and also we have*

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha.$$

(b)Theorem (Linearity on integrator)

If $f \in R(\alpha), f \in R(\beta)$ *on* $[a, b]$, *then* $f \in R\,(c_1\alpha + c_2\beta)$ *on* $[a, b]$, *where* c_1 *and* c_2 *are two constants and also we have*

$$\int_a^b f\, d\,(c_1\alpha + c_2\beta) = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta.$$

(c)Theorem (Additive property of R-S integral)

Assume that $c \in (a, b)$. *If any two of the three integrals in* (1) *exist, then the third also exists and we have*

$$\int_a^b f d\alpha + \int_a^b f d\alpha = \int_a^b f d\alpha \quad ...(1)$$

6.4 Formula for Integration by Parts [1 hr]

(a) Theorem (Formula for integration by parts)**:**

If $f \in R(\alpha)$ *on* $[a, b]$, *then* $\alpha \in R(f)$ *on* $[a, b]$. *Moreover, we have*

$$\int_a^b f(x)\, d\alpha(x) + \int_a^b \alpha(x)\, df(x) = f(b)\, \alpha(b) - f(a)\, \alpha(a).$$

(b) Example to support above formula.

6.5 Change of Variables in R-S integrals **[1 hr]**

(a) Theorem 6.5 (R-S Integrability of Composite Functions):

(i) *Let* $f \in R(\alpha)$ *on* $[a, b]$ *and let* g *be a strictly monotonic continuous function defined on interval* $[c, d]$.

(ii) *Let* $a = g(c)$ *and* $b = g(d)$.

(iii) *Let* h *and* β *be the composite functions defined by*

$$h(x) = f[g(x)] \text{ and } \beta(x) = \alpha[g(x)] \quad \forall x \in S.$$

Then $h \in R(\beta)$ *on* S *and we have* $\int_a^b f d\alpha = \int_c^d h d\beta$.

$$\text{i.e.} \qquad \int_{g(c)}^{g(d)} f(x)\, d\alpha(x) = \int_c^d f[g(x)]\, d\{\alpha[g(x)]\}.$$

6.6 Reduction to a Riemann Integral. **[1 hr]**

(a)Theorem: (Relation between R-integral and RS-integral)

Let $f \in R(\alpha)$ *on* $[a, b]$ *and assume that* α *has a continuous derivative* α' *on* $[a, b]$. *Then, the Riemann integral* $\int_a^b f(x)\alpha'(x)\, dx$ *exists and we have*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\alpha'(x)\, dx.$$

In other words, R-S integral of f *w.r.t.* α *on* $[a, b]$ *is equal to the Riemann integral of* $f\alpha'$ *on* $[a, b]$.

(b) Example to support above Theorem.

6.7 Step function as Integrators. **[1.5 hrs]**

(a) Definition of Step Functions

(b) Theorem (Step functions as integrator)

Let $c \in (a, b)$ *and define* α *on* $[a, b]$ *as follows. The values of* $\alpha(a)$, $\alpha(c)$, $\alpha(b)$ *are arbitrary and* $\alpha(x) = \alpha(a)$ *if* $a \le x < c$ and $\alpha(x) = \alpha(b)$ *if* $c < x \le b$.

Let f *be defined on* $[a, b]$ *in such a way that at least one of* f *or* α *is continuous at* c *from the left and at least one is continuous at* c *from the right at* c.

Then $f \in R(\alpha)$ *on* $[a, b]$ *and we have* $\int_a^b f d\alpha = f(c)\,[\alpha(c+) - \alpha(c-)]$.

(c) Examples:

i. *The existence of the value of R-S integral can also be affected by changing the value of the integrand* f *at a single point.*

ii. *In a Riemann integral* $\int_a^b f(x)\, dx$**,** *the values of* f *can be changed at finite number of points without affecting either the existence or the value of the integral.*

****6.8 Reduction of R-S Integral to a Finite sum.** **[0.5 hrs]**

(a) Theorem (Reduction of a R-S integral to a finite sum) **(Statement Only)**

(b) Theorem (Reduction of finite Sum to R-S integral). **(Statement Only)**

6.9 Monotonically Increasing Integrators, Upper & Lower Integrals. **[2 hrs]**

(a) Definition of upper and lower Stieltjes sums and their integrals.

(b) Theorem (Refinement of partition):

Assume that $\alpha \uparrow$ *on* $[a, b]$. *Then*

(i) $P, P' \in P[a, b]$ *and* $P' \supseteq P \Rightarrow U(P', f, \alpha) \leq U(P, f, \alpha)$ *and* $L(P', f, \alpha) \geq L(P, f, \alpha)$

(ii) *For any two partitions* P_1 *and* P_2 *of* $[a, b]$, $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

(c) Theorem (Comparison of Lower and Upper Integrals):

Assume that $\alpha \uparrow$ on $[a, b]$. *Then* $\underline{I}(f, \alpha) \leq \bar{I}(f, \alpha)$.

(d) Example to show $\underline{I}(f, \alpha) \leq \bar{I}(f, \alpha)$.

6.10 Riemann's Conditions *for* R-S Integral. **[1 hr]**

(a) Definition of Riemann's condition.

(b) Theorem (Riemann's condition)

6.11 Comparison Theorems. **[2 hrs]**

(a) Theorem (Order preserving property of R-S integral)

Assume that $\alpha \uparrow$ *on* $[a, b]$, $f, g \in R(\alpha)$ *on* $[a, b]$ *and if* $f(x) \leq g(x)\ \forall x \in [a, b]$,

then we have $\int_a^b f(x)\, d\alpha(x) \leq \int_a^b g(x)\, d\alpha(x)$

(b) Theorem: *Assume that* $\alpha \uparrow$ *on* $[a, b]$ *and* $f \in R(\alpha)$ on $[a, b]$, *then* $|f| \in R(\alpha)$ on $[a, b]$ *and we have*

$$\left| \int_a^b f(x)\, d\alpha(x) \right| \leq \int_a^b |f(x)|\, d\alpha(x).$$

(c) Theorem (R-S integrability of the square function):

Assume that $\alpha \uparrow$ *on* $[a, b]$ *and* $f \in R(\alpha)$ *on* $[a, b]$, *then* $f^2 \in R(\alpha)$ *on* $[a, b]$.

(d) Theorem (R-S Integrability of the product):

Assume that $\alpha \uparrow$ *on* $[a, b]$. *If* $f \in R(\alpha)$ *on* $[a, b]$ *and* $g \in R(\alpha)$ *on* $[a, b]$, *then the product* $f \cdot g \in R(\alpha)$ *on* $[a, b]$.

In other words, the product of two R-S integrable functions is again R-S integrable.

6.12 Necessary and Sufficient Conditions. **[2 hrs]**

(a)Theorem (Sufficient condition for R-S integrals):

If f *is continuous on* $[a, b]$ *and if* α *is of BV on* $[a, b]$, *then* $f \in R(\alpha)$ *on* $[a, b]$.

(b) Theorem (Sufficient condition for R-S integrals):

Let f *be of BV on* $[a, b]$ *and* α *is continuous on* $[a, b]$, *then* $f \in R(\alpha)$ *on* $[a, b]$.

(c)Theorem (Sufficient condition for Riemann integrals):

Each of the following conditions is sufficient for the existence of the Riemann integral $\int_a^b f(x)dx$:

(i) f *is continuous on* $[a, b]$ (ii) f *is of BV on* $[a, b]$.

(d)Theorem (Necessary condition for existence of R-S integrals):

Assume that $\alpha \uparrow$ *on* $[a, b]$ *and* $c \in (a, b)$. *Assume that both* f *and* α *are discontinuous at* $x = c$ *from the right i.e. assume that there exists an* $\varepsilon > 0$ *such that* $\forall\delta > 0$, *there are values of x and y in the interval* $(c, c + \delta)$ *satisfying*

$$| f(x) - f(c) | \geq \varepsilon \text{ and } | \alpha(y) - \alpha(c) | \geq \varepsilon.$$

Then the integral $\int_a^b f(x)\, d\alpha(x)$ *cannot exist. Similarly, if both f and* α *are discontinuous at* $x = c$ *from the left, then* $\int_a^b f(x)dx$ *cannot exist.*

6.13 Mean Value Theorem (MVT) for R-S Integrals. [2 hrs]

(a) Theorem (First mean value theorem (FMVT) for R-S integrals):

Assume that $\alpha \uparrow$ *on* $[a, b]$ *and* $f \in R(\alpha)$ *on* $[a, b]$. *Let M and m are the supremum and infimum of the set* $\{f(x) : x \in [a, b]\}$. *Then* $\exists$ *a real number c satisfying* $m \leq c \leq M$ *such that* $\int_a^b f(x)\, d\alpha(x) = c \int_a^b d\alpha(x) = c[\alpha(b) - \alpha(a)]$.

In particular, if f is continuous on $[a, b]$, *then* $c = f(x_0)$ *for some* x_0 *in* $[a, b]$.

(b) *Show that the restriction that c is a point of the closed interval is essential in the statement of* FMVT.

(c)Theorem (Second mean value theorem for R-S integral):

Assume that α *is continuous and* $f \uparrow$ *on* $[a, b]$. *Then* $\exists$ *a point* $x_0 \in [a, b]$ *such that*

$$\int_a^b f(x)\, d\alpha(x) = f(a) \int_a^{x_0} d\alpha(x) + f(b) \int_{x_0}^b d\alpha(x).$$

(d) Statement of the corresponding theorem for Riemann Integral.

6.14 Integral as the Function of the Integral. [1.5 hr]

(a) Definition of integral function

(b) Theorem: *Let* α *be of BV on* $[a,b]$ *and assume that* $f \in R(\alpha)$ *on* $[a, b]$. *Define F on* $[a, b]$ *by an equation*

$$F(x) = \int_a^x f(x)\, d\alpha(x), \text{ where } x \in [a, b]. \text{ Then we have}$$

(i) *F is of bounded variation on* $[a, b]$.

(ii) *Every point of continuity of* α *is also a point of continuity of F.*

(iii) *If* $\alpha \uparrow$ *on* $[a, b]$, *then the derivative of* $F(x)$ *exists at point* $x \in (a, b)$, *where* $\alpha'(x)$ *exists and f is continuous. For such x, we have* $F'(x) = f(x)\, \alpha'(x)$.

(c) Review (without proof) of Theorem of Integrators of BV.

(d) Theorem (Conversion a Riemann integral of the product $f.g$ into a R– S integral):

Let $f \in R$ *on* $[a, b]$, $g \in R$ *on* $[a, b]$ *and let*

$F(x) = \int_a^x f(t)dt,\ \ G(x) = \int_a^x g(t)dt$ *if* $x \in [a, b]$.

Then F and G are continuous functions of bounded variation on $[a, b]$. *Also* $f \in R(G)$ *and* $g \in R(F)$ *on* $[a, b]$, *and we have*

$$\int_a^b f(x)\, g(x)\, dx = \int_a^b f(x)\, dG(x) = \int_a^b g(x)\, dF(x).$$

6.15 Fundamental Theorem of Integral Calculus. **[1.5 hr]**

(a) Definition of Primitive (or antiderivative) of a function with examples:

(b) Theorem (First fundamental theorem of integral calculus):

Let $f \in R$ *and be continuous on* $[a, b]$. *Let* $F(x) = \int_a^x f(t)\, dt$ *for all* $x \in [a, b]$, *then the derivative* $F'(x)$ *exists and* $F'(x) = f(x)$.

In other words, the integral function $F(x)$ *of a continuous function f is differentiable for all* $x \in [a, b]$ *and satisfies* $\frac{d}{dx}[F(x)] = f(x)$.

(c) Theorem (Second fundamental theorem of integral calculus):

Assume that $f \in R$ *on* $[a, b]$ *and let F be a function defined on* $[a, b]$ *such that the derivative* $F'(x)$ *exists at each point* $x \in (a, b)$ *and* $F'(x) = f(x),\ \forall\, x \in (a, b)$.

At the end points, assume that $F(a+)$ *and* $F(b-)$ *exist and satisfy*

$$F(a) - F(a+) = F(b) - F(b-).$$

Then we can have $\int_a^b f(x)\, dx = \int_a^b F'(x)\, dx = F(b) - F(a)$.

In other words, if $F : [a, b] \to \mathbb{R}$ *is a primitive of* $f : [a, b] \to \mathbb{R}$, *then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Exercises. **[2 hrs]**

At least the following problems from Exercise on page 174, Chapter-7 of the book *Mathematical Analysis*, Tom Apostol II Edition (1987) should be discussed:

7.1, 7.2, 7.3, 7.11, 7.12, 7.13.

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UNIT 7 - SEQUENCES AND SERIES OF FUNCTIONS [19 HRS]

7.1 Sequence of Functions. **[6 hrs]**

(a) Introduction of sequences of functions.

(b) Definition of limit function and pointwise convergence.

(c) Examples:

- *sequence of continuous functions with a discontinuous limit function.*
- *sequence of function for which* $\lim_{n\to\infty} \int_0^1 f(x)\, dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\, dx$
- *Sequence of differentiable functions* $\{f_n\}$ *with limit* '0' *for which* $\{f_n'\}$ *diverges.*

(d) Definition of uniform convergence and uniform boundedness.

(e) Geometrical interpretation of uniform convergence.

(f) Proof of the fact that uniform convergence of a sequence of functions implies its pointwise convergence but converse may not be true.

(g) Cauchy criterion for uniform convergence for sequence and its proof.

(h) Theorem related to uniform convergence and continuity:

(i) Theorem (Uniform convergence preserves continuity**):** *If* $f_n \to f$ *uniformly on a set* $S \subseteq \mathbb{R}$, *and if each* f_n *is continuous on* S, *then* f *is continuous on* S.

(ii) Theorem (Dini's uniform convergence theorem):

If $\{f_n\}$ is a *sequence of continuous functions converging point wise to a continuous function* f *on a compact set* $S \subseteq \mathbb{R}$, *and if,* $\forall x \in S$, $\{f_n(x)\}$ *is monotone decreasing, then* $f_n \to f$ (*uniformly*) *on* S .

(i) Uniform convergence and differentiation for sequence

Theorem: *Assume that* $\{f_n\}$ *is a real valued function having a finite derivative at each point of* (a, b). *Assume that for at least one point* x_0 *in* (a, b), *the sequence* $\{f_n(x_0)\}$ *converges. Assume further that there exists a function g such that* $f_n' \to g$ *uniformly on* (a, b). *Then*

(i) *there exists a function* f *such that* $f_n \to f$ *uniformly on* (a, b).

(ii) *for each x in* (a, b), *the derivative* $f'(x)$ *exists and* $f'(x) = g(x)$.

(j) Uniform convergence and integration for sequence:

Theorem: *Let* $\{f_n\}$ *be a sequence of continuous functions on* $[a, b]$. *If* $f_n \to f$ *uniformly on* $[a, b]$, *then* $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$

7.2 Series of Functions. [7 hrs]

(a) Definition of uniform convergence of series of functions.

(b) The theorem of Cauchy condition for uniform convergence of series and its poof.

(c) Proof of Weierstrass M-test, Dirichlet's test, Abel's test, and their applications to test the uniform convergence of series.

(d) The theorem on continuity of the limit function of a uniformly convergent series.

Theorem: *Assume that* $\sum f_n(x) = f(x)$ (*uniformly on S*). *If each* f_n *is continuous at a point c of S, then* f *is also continuous at c.*

(e) Uniform convergence and differentiation of Series
Theorem*:* (Without Proof)

Assume that each f_n *is a real valued function defined on* (a, b) *such that the derivative* $f_n'(x)$ *exists for each* $x \in (a, b)$. *Assume that for at least one point* x_0 *in* (a, b), *the series* $\sum f_n(x_0)$ *converges. Assume further that there exists a function g such that* $\sum f_n'(x) = g(x)$ (*uniformly*) *on* (a, b). *Then*

(i) *there exists a function* f *such that* $\sum f_n(x) = f(x)$ (*uniformly*) *on* (a, b).

(ii) *if* $x \in (a, b)$, *the derivative* $f'(x)$ *exists and equals to* $\sum f_n'(x)$.

(f) Uniform convergence and integration for series:

Theorem (*Term by term Integration of Series):*

Let $\sum_{n=1}^{\infty} f_n$ *be a series of continuous functions on* $[a, b]$. *If* $\sum_{n=1}^{\infty} f_n$ *converges uniformly to* f *on* $[a, b]$, *then* $\sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx \ \forall x \in [a, b]$.

Exercises. **[3 hrs]**

At least the following problems from Exercise on page 247, Chapter-9 of the book *Mathematical Analysis*, Tom M. Apostol II Edition (1987) should be discussed:

9.1, 9.3, 9.4, 9.5, 9.6, 9.11, 9.12, 9.13, 9.18, 9.22

Note: For this chapter 7; 3 lecturer hrs should be taken from hours allotted for review.

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UNIT 8 - IMPROPER INTEGRALS. [18 HRS]

8.1 Classification of Improper Integrals into two kinds with examples. [1 hr]

8.2 Improper Integral of First Kind. [9 hrs]

(a) Definition of convergence and divergence

(b) Application of the fundamental theorem of integral calculus.

(c) The convergence or the divergence of the improper integrals of the functions:

i) $1/x^2$ on $(1, \infty)$ ii) $\sin x$ on $(0, \infty)$

iii) $1/x^{\alpha}$ on (a, ∞), where $a > 0$ iv) e^{-rx} on $(0, \infty)$, where $r > 0$.

(d) Geometrical interpretation of Convergent improper integral of the first kind

(e) Analogy of the improper integral $\int_a^{\infty} f(x)\,dx$ with infinite series $\sum_{n=1}^{\infty} a_n$.

(f) The theorem on the Cauchy criterion for convergence.

Theorem: (Cauchy Criterion): *Let f be integrable over* $[a, t]$ *for all* $t \geq a$. *Then the integral* $\int_a^{\infty} f(x)\,dx$ *converges if, and only if, for every* $\varepsilon > 0$, *there exists a point* $t_0 > a$ *such that for any two points* t'', t' *satisfying* $t'' > t' \geq t_0$ *implies* $\left|\int_{t'}^{t''} f(x)\,dx\right| < \varepsilon$.

(g) Linear properties of improper integral:

1. *If* $\int_a^{\infty} f(x)\,dx$ *and* $\int_a^{\infty} g(x)\,dx$ *exist, then* $\int_a^{\infty} [f(x) \pm g(x)]\,dx$ *exists and we have*

$$\int_a^{\infty} [f(x) \pm g(x)]\,dx = \int_a^{\infty} f(x)\,dx \pm \int_a^{\infty} g(x)\,dx.$$

2. *If* $\int_a^{\infty} f(x)\,dx$ *exists, then for every* $c \in \mathbb{R}$, $\int_a^{\infty} c\,f(x)\,dx$ *exists and*

$$\int_a^{\infty} c\,f(x)\,dx = c\int_a^{\infty} f(x)\,dx.$$

(h) A necessary and sufficient condition for the convergence of the improper integral:

Theorem: *Let* $f(x) \geq 0$ *for all* $x \geq a$ *and let f be integrable over* $[a, t]$ *for all* $t \geq a$.

Then the integral $\int_a^{t} f(x)\,dx$ *converges if, and only if, the set* $\left\{I_t = \int_a^{t} f(x)\,dx,\ t \geq a\right\}$ *is bounded. In this case* $\int_a^{\infty} f(x)\,dx = \sup\left\{I_t : I_t = \int_a^{t} f(x)\,dx,\ t \geq a\right\}$.

(i) The comparison test, the limit comparison test to determine the convergence or divergence of some improper integrals.

Theorem: (**Comparison test for convergence and divergence**).

Let f and g be two integrable functions over $[a, t]$ *for all* $t \geq a$ *and if* $0 \leq f(x) \leq g(x)$ *for all* $x \geq a$. *Then*

(i) *If the integral* $\int_a^\infty g(x)\,dx$ *converges, then the integral* $\int_a^\infty f(x)\,dx$ *also converges.*

(ii) *If the integral* $\int_a^\infty f(x)\,dx$ *diverges, then the integral* $\int_a^\infty g(x)\,dx$ *also diverges.*

Theorem: (Limit comparison test): *Let* $g(x) > 0$ *and if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$,

finite and nonzero, then both integrals converge or diverge together.

Also if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ *and* $\int_a^\infty g(x)\,dx$ *converges, then the integral* $\int_a^\infty f(x)\,dx$ *also converges, and if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} \to \infty$ *and* $\int_a^\infty g(x)\,dx$ *diverges, then* $\int_a^\infty f(x)\,dx$ *also diverges.*

Theorem: (Limit Comparison Test for Convergence)

Let $0 \leq f$ *be integrable over* $[a, t]$ *for all* $t \geq a$ *and for* $p > 1$, $\lim_{x\to\infty} x^p f(x) = L$ (*finitely exists*), *then the integral* $\int_a^\infty f(x)\,dx$ *converges.*

Theorem: *Let* $f(x)$ *be integrable over* $[a, t]$ *for all* $t \geq a$ *and* $\lim_{x\to\infty} x f(x) = L \neq 0$ ($or = \pm\infty$). *Then the integral* $\int_a^\infty f(x)\,dx$ *diverges.*

Theorem: (Statement Only) *Let f be integrable over* $[a, t]$, $\forall t \geq a$ *and*

$\lim_{x\to\infty} x f(x) = L$ *or* $\pm\infty$. *If* $L \neq 0$, *then* $\int_a^\infty f(x)\,dx$ *diverges and if* $L = 0$, *the test fails.*

(j) Definition of absolute and **Conditionally** convergence of $\int_a^\infty f(x)$

(k) The theorem asserting that absolute convergence implies its convergence.

Theorem: *If the integral* $\int_a^\infty |f(x)|\,dx$ *converges, then* $\int_a^\infty f(x)\,dx$ *also converges.*

Every absolutely convergent improper integral is necessarily convergent.

(l) The theorem on absolute convergence of the improper integral of a product .

Theorem: (Absolutely Integrable of the Product)

If $\int_a^\infty f(x)\,dx$ *converges absolutely and g is bounded on* $[a, \infty)$, *then the product* $f(x)\,g(x)$ *is also absolutely integrable over* $[a, \infty)$.

(m) Dirichlet's test and Abel's test for the convergence of the improper integral of a product

Theorem: (Dirichlet's Test): *Assume that*

(i) *f is integrable over $[a, t]$ for all $t \geq a$ and there exists a constant $M > 0$ such that* $\forall t \geq a$ $\left| \int_a^t f(x)\, dx \right| \leq M$,

(ii) *$g(x)$ is monotonic decreasing to 0 as $x \to \infty$ i.e. $g(x) \to 0$ as $x \to \infty$.*

Then the integral $\int_a^\infty f(x)\, g(x)\, dx$ is convergent.

Theorem (Abel's Test): *Assume that*

(i) *The integral $\int_a^\infty f(x)\, dx$ converges*

(ii) *$g(x)$ is monotonic and bounded on $[a, \infty)$ i.e. $\exists$ a constant $M > 0$ such that $|g(x)| \leq M$, $\forall x \geq a$, then the integral $\int_a^\infty f(x)\, g(x)\, dx$ is convergent.*

(n) Variants of the improper integral of the first kind.

(o) Definition of the Cauchy principle value of the integral over $\mathbb{R}$.

To show the fact that the existence of the Cauchy principal value may not always imply the convergence of the integral $\int_{-\infty}^{\infty} f(x)\, dx$.

8.3 Improper Integrals of Second Kind. [5 hrs]

(a) Definition of convergence and divergence.

(b) Reduction of the improper integral of the second kind into that of the first kind.

(c) Geometrical interpretation of the convergent improper integral of Second Kind $\int_a^b f(x)$.

(d) Theorem on the Cauchy criterion for convergence.

Theorem: (Cauchy Criterion) (Without Proof)

Assume that f be integrable over $[t, b]$ for all $t \in (a, b]$. Then the integral $\int_a^b f(x)\, dx$ converges if, and only if, for every $\varepsilon > 0$, there exists $t_0 \in (a, b]$ such that $a < t' \leq t'' \leq t_0$ implies

$$\left| \int_{t'}^{b} f(x)\, dx - \int_{t''}^{b} f(x)\, dx \right| = \left| \int_{t'}^{t''} f(x)\, dx \right| < \varepsilon.$$

(e) **The comparison test, the limit comparison test** and apply them to determine convergence or divergence of some improper integrals.

Theorem: (Comparison test for Integrands) (Without Proof)

Let f, g be two integrable functions over $[t, b]$ for all $t \in (a, b]$ and if $0 \leq f(x) \leq g(x)$ for all $x \in (a, b]$. Then

(i) *If the integral $\int_a^b g(x)\, dx$ converges, then the integral $\int_a^b f(x)\, dx$ also converges.*

(ii) *If the integral $\int_a^b f(x)\, dx$ diverges, then the integral $\int_a^b g(x)\, dx$ also diverges.*

Remarks: Above comparison test can be formulated in limit form as follows:

Theorem: (Comparison test for Integrands in Limit) (Without Proof)

Let f, g be two functions integrable over $(t, b]$ *for all* $t \in (a, b]$ *and if* $\lim_{x \to a+} \frac{f(x)}{g(x)} = L$,

where L is finite. Then the two integrals $\int_a^b f(x)\,dx$ *and* $\int_a^b g(x)\,dx$ *converge and diverge together.*

Theorem: (Limit Comparison Test for Convergence and Divergence) (Without Proof)

Let $0 \le f(x)$ *be bounded and integrable over* $[t, b]$ *for all* $t \in (a, b]$. *Then*

(i) *If* $0 < p < 1$ *and* $\lim_{x \to a+} (x - a)^p f(x)$ *exists and say L, then* $\int_a^b f(x)\,dx$ *converges.*

(ii) *If* $p \ge 1$ *and* $\lim_{x \to a+} (x - a)^p f(x)$ *exists and non zero* $(\pm \infty)$, *then the integral diverges.*

(f) The variants of the improper integral of the second kind.

(g) Definition of the Cauchy principle value of the improper integral of the second kind.

Example to show that the existence of the Cauchy principal value may not imply the convergence of the integral.

Exercises. **[4 hrs]**

From the book "Advanced Calculus by D.V. Widder", Chapter 10, related problems should be discussed.

Note: If time permits, one can give a detailed theories and list of simple problems for exercise to the students from other books like **'Mathematical Analysis'** by *S.C. Malik and Savita Arora*, 4th Edition Chapter 10, **'Real Analysis'** by *Dipak Chatterjee*, chapter -12, etc.

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UNIT 9. COMPLEX NUMBERS AND FUNCTIONS [10 LECTURES]

9.1 Review, Different forms and Roots of complex numbers.

(a) Review of definition of complex number, Sum and product of complex numbers, Basic algebraic properties of complex numbers, Modulus and conjugate of complex numbers, Triangle Inequality, Polar form of complex numbers, Related Examples. **[2 Hours]**

(b) Products, Quotients and Powers in Exponential form, Arguments of products and quotients.

Example 1, 2 ; Page No. 20 and Example 1 , 2 Page No. 21, 22 **[2 Hours]**

(c) Roots of complex numbers

Example 1, 2 & 3, Page No. 27, 28, 29 **[2 Hours]**

(d) Region in the complex plane:

Definition of ε - neighbourhood, Deleted neighbourhood, Interior point, exterior point and boundary point, Open and closed sets, Domain and region, Bounded and unbounded set. **[1 Hour]**

9.2 Function of a complex variable and mappings **[3 Hours]**

(a) Definition of function of a complex variable.

(b) Definition of mapping with examples.

` (c) Mapping by $w = z^2$. Example 1, on Page No. 39

(d) Mapping by $w = e^z$, Example 1 & 2 on Page No. 42 and 43.

Exercises. **[2 hrs]**

At least the following problems from the following pages of the book ***Complex Variables and Applications, 8th edition*, by James Ward Brown & Ruel V. Churchill** should be discussed:

Page No. 5 : 1, 2, 3, 4; Page No. 8 : 1, 2, 3 ; Page No. 12: 1, 3, 4, 5, 6
Page No. 14: 1, 2, 7, 13, 15 ; Page No. (22-23): 1, 2, 3, 5, 6; Page No. (29-30): 1, 2, 3, 8
Page No. 37: 1, 2, 3.

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UNIT 10. ANALYTIC FUNCTIONS [16 LECTURES]

10.1 Limits and continuity

(a) Definition of limit of function of a complex variable $f(z)$ as $z \to z_0$
Examples 1 and 2, Page No. (46 - 47).

(b) Theorems on Limits:

(i) Theorem: *Suppose that* $f(z) = u(x, y) + i\, v(x, y)$ $(z = x + iy)$
and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$.
Then $\lim_{z\to z_0} f(z) = w_0$ *if and only if*
$\lim_{(x, y) \to (x_0, y_0)} u(x, y) = u_0$ *and* $\lim_{(x, y) \to (x_0, y_0)} v(x, y) = v_0.$

(ii) Theorem: (Without proof): *Suppose that* $\lim_{z\to z_0} f(z) = w_0$ *and* $\lim_{z\to z_0} F(z) = W_0$
Then $\lim_{z\to z_0} [f(z) + F(z)] = w_0 + W_0$; $\lim_{z\to z_0} [f(z)\ F(z)] = w_0\, W_0$;
and if $W_0 \neq 0$, $\lim_{z\to z_0} \dfrac{f(z)}{F(z)} = \dfrac{w_0}{W_0}$.

(c) Limit involving the points at infinity.

Theorem: *If z_0 and w_0 are points in the z and w planes respectively, then*

$$\lim_{z\to z_0} f(z) = \infty \text{ if and only if } \lim_{z\to z_0} \frac{1}{f(z)} = 0$$

and $$\lim_{z\to\infty} f(z) = w_0 \text{ if and only if } \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0.$$

Moreover, $$\lim_{z\to\infty} f(z) = \infty \text{ if and only if } \lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0.$$

Examples on Page No. 52.

(d) Continuity:

Definition of continuity of function of a complex variable $f(z)$ as z at z_0

Theorem: *The composition of continuous functions is itself continuous*.

Theorem: *If a function $f(z)$ is continuous and non-zero at a point z_0, then $f(z) \neq 0$ throughout some neighbourhood of that point*.

10.2 Complex Differentiation

(a) Definition of derivative of a function of complex variable.
Examples 1, 2, 3 on Page No. 57,58.

(b) Differentiation formulas
(**without proof,** as its proof is similar to those in real variables).

(c) Theorem: *Suppose that $f(z) = u(x, y) + i\, v(x,y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + i\, y_0$. Then the first order partial derivatives of u and v must exists at (x_0, y_0) and satisfy the Cauchy Riemann equations* $u_x = v_y$ and $u_y = -v_x$

Also, $f'(z_0) = u_x + i\, v_x$, *where these partial derivatives are to be evaluated at* (x_0, y_0).

Examples 1 and 2, on Page No. 65, 66.

(d) Sufficient condition for differentiability

Theorem: *Let* $f(z) = u(x, y) + i\, v(x, y)$ *be defined throughout some* ε- *neighbourhood of a point* $z_0 = x_0 + i\, y_0$, *and suppose that*

(i) *the first order partial derivatives of the functions u and v with respect to x and y exists everywhere in the neighborhood*;

(ii) *those partial derivatives are continuous at* (x_0, y_0) *and satisfy the Cauchy – Reimann equations* $u_x = v_y$ *and* $u_y = -v_x$ *at* (x_0, y_0).

Then $f'(z_0)$ *exists and its value being* $f'(z_0) = u_x + i\, v_x$ *where the right hand side is to be evaluated at* (x_0, y_0).

Examples 1,2 of Page No. 68

(e) Cauchy – Reimann Equation in polar form

Theorem: *Let the function* $f(z) = u(r, \theta) + i\, v(r, \theta)$ *be defined throughout some* ε- *neighbourhood of a non zero point* $z_0 = r_0\, e^{i\theta_0}$ *and suppose that*

(i) *the first order partial derivatives of the function u and v with respect to r and* θ *exists everywhere in the neighborhood*;

(ii) *those partial derivatives are continuous at* (r_0, θ_0) *and satisfy the polar form* $ru_r = v_\theta$ and $u_\theta = -rv_r$ of Cauchy Riemann equations at (r_0, θ_0).

Then $f'(z_0)$ exists, its value being $f'(z_0) = e^{-i\theta}(u_r + iv_r)$

where R.H.S. is to be evaluated at (v_0, θ_0)

Examples 1, Page No. 70

10.3 Analytic Functions

(a) Definition of Analytic functions, Entire function, Singular points, Relations between continuity, differentiability and analyticity.

Examples 1, 2, 3, Page No. 75-76.

(b) Definition of Harmonic function and Harmonic conjugate

Theorem: *If a function* $f(z) = u(x, y) + i\, v(x, y)$ *is analytic in a domain D, then its component functions u and v are harmonic in D.*

Theorem: *A function* $f(z) = u(x, y) + i\, v(x, y)$ *is analytic in a domain D if and only if v is a harmonic conjugate of u.*

Example 3, 4, 5 Page No. (80-81).

(c) Reflection Principle

Theorem: *Suppose that a function f is analytic in some domain D which contains a segment of the x-axis and whose lower half is the reflection of the upper half with respect to that axis. Then* $\overline{f(z)} = f(\bar{z})$ *for each point z in the domain if, and only if ,* $f(x)$ *is real for each point x on the segment.*

Examples on Page No. 97.

Exercises. **[2 hrs]**

At least the following problems from the following pages of the book ***Complex Variables and Applications, 8th edition**, **by James Ward Brown & Ruel V. Churchill*** should be discussed:

Page No. 55 : Q.No.1, 2, 3 ; Page No. 62 : Q.No. 1; Page No. 71: Q. No. 1, 2, 3; Page No. 77: Q.No.1, 2, 4; Page No. 81: Q.No. 1, 2.

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MODEL QUESTION

Tribhuvan University

Level: 4 years Bachelor / Sci. & Tech. / IV Year **Full Marks: 100**

Subject: Mathematical Analysis II (MAT. 402) **Time : 3 hrs**

Pass Marks: 35

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt ALL the questions.

1. Distinguish between accumulation points and adherent points of a set with examples. Show that if x is an accumulation point of $S \subseteq \mathbb{R}^n$, then every open n-ball $B(\text{x})$ contains infinitely many points of S. Also prove that a set S in $\mathbb{R}^n$ is closed if, and only if, it contains all its adherent point. **[3 + 3 + 4]**

OR

Define a metric space. If (M, d) is a metric space, define $d'(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$, show that d' is also a metric for M. Let (S, d) be a metric subspace of a metric space (M, d) and $X \subseteq S$. Prove that X is open in S if, and only if, $X = A \cap S$ for some set A which is open in M. State corresponding theorem for closed set. **[1 + 3 + 5 +1]**

2. Define the terms: covering, open covering and compact set in $\mathbb{R}^n$ with example. Show that $S = (0, 1)$ in $\mathbb{R}^1$ is not compact. Prove that a set S in $\mathbb{R}^n$ is compact if, and only if, it is closed and bounded in $\mathbb{R}^n$. **[3 + 1 + 6]**

3. What do you mean by the complete metric space? Prove that every Euclidean space $\mathbf{R}^k$ is complete. Is this theorem equally hold in an arbitrary metric space? Justify.

Let $A = \mathbb{R}\text{-}\{0, -2\}$, define $f : A \to \mathbb{R}$ by $f(x) = \dfrac{x}{2x + x^2}$.

Use sequential criterion to show that $\lim\limits_{x \to 0} f(x) = \dfrac{1}{2}$. **[1 + 5 + 2+ 2]**

4. Define the total derivative of a function from $\mathbf{R}^n$ to $\mathbf{R}^m$. Let $\mathbf{S}$ be a subset of $\mathbb{R}^n$. If $\mathbf{f} : \mathbf{S} \to \mathbb{R}^m$ is differentiable at $\mathbf{c}$, then prove that $\mathbf{f}$ is continuous at $\mathbf{c}$. Show that the existence of finite directional derivative $\mathbf{f}'(\mathbf{c}, \mathbf{u})$ of a function $\mathbf{f}$ at $\mathbf{c}$ in all the direction given by $\mathbf{u}$ may not imply the continuity of that function at $\mathbf{c}$. **[2 + 3 + 5]**

OR

State the Young's theorem for sufficient condition for the equality of the mixed partial derivatives. Show that $D_{1,2} f(0, 0) \neq D_{2,1} f(0, 0)$ i.e. $f_{xy} \neq f_{yx}$ for the function

$f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} &, \; if \; (x, y) \neq (0, 0) \\ 0 &, \; if \; (x, y) = (0, 0). \end{cases}$ **[2+5]**

5. Define a function of bounded variation on [a, b]. Show that the function f defined by

$$f(x) = x^2 \sin(1/x) \text{ if } x \neq 0 \text{ and } f(0) = 0$$

is of bounded variation on the interval [0, 1]. Let each function f and g be of bounded variation on [a, b]. Prove or disprove that $f + g$ and $\dfrac{1}{f}$ are of bounded variation on [a, b].

[2 + 4 + 4]

6. When is a bounded function f said to be integrable with respect to α on an interval $[a, b]$? If $f \in R(\alpha)$ on $[a, b]$, then prove that $\alpha \in R(f)$ on $[a, b]$ and prove that

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df(x) = f(b)\,\alpha(b) - f(a)\,\alpha(a).$$

Using this result, evaluate $\int_0^{\pi/2} x\, d(\cos 2x)$. **[1 + 7 + 2]**

OR

State Riemann's condition for integrability of f w.r.t. α on $[a, b]$. Assume that $\alpha \uparrow$ on $[a, b]$ and $f \in R(\alpha)$ on $[a, b]$, then prove that $|f| \in R(\alpha)$ on $[a, b]$. Let $\alpha \uparrow$ on $[a, b]$ and let $c \in (a, b)$. Prove that $\int_a^b f\,d\alpha$ cannot exist if both f and α are discontinuous from the left at point $x = c$. **[1 + 5 +4]**

7. Define uniform convergence of a sequence of functions on a set and interpret it geometrically. By taking the sequence $f_n(x) = n^2 x\,(1 - x)^n$, $n = 1, 2, 3, \ldots$, verify that the operation of 'limit' and 'integration' can not be interchanged.

i.e. $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim\limits_{n\to\infty} f_n(x)dx$. **[2 + 2 +6]**

8. Assume that f is integrable over $[a, t]$ for all $t \ge a$ and there exists a constant $M > 0$ such that $\forall t \ge a$, $\left|\int_a^t f(x)\,dx\right| \le M$. Further suppose that $g(x)$ is monotonically decreasing and $g(x) \to 0$ as $x \to \infty$. Prove that the integral $\int_a^\infty f(x)\,g(x)\,dx$ is convergent.

Using this convergence criterion, show that the integral $\int_a^\infty \frac{\sin x}{x^p}\,dx$ is convergent for all $x \ge a > 0$ and $p > 0$. **[6 + 4]**

9. a) Define n^{th} roots of a complex number. Find all the values of $(8 - 8\sqrt{3}\,i)^{1/4}$ and identify which is the principal root. **[1 + 3 +1]**

b) Define the mapping $w = z^2$. Find and sketch the images of the hyperbolas $x^2 - y^2 = c_1$ ($c_1 > 0$) and $2xy = c_2$ ($c_2 > 0$) under the transformation $w = z^2$. **[1 + 4]**

10. Suppose that $f(z) = u(x, y) + i\,v(x, y)$ and that $f'(z_0)$ exists at a point $z_0 = x_0 + i\,y_0$. Prove that the first order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy – Reimann conditions $u_x = v_y$ and $u_y = -v_x$ there.
Also prove that $f'(z_0) = u_x + i\,v_x$,where these partial derivatives are to be evaluated at (x_0, y_0).

Using this theorem, show that $f'(z)$ does not exist at any point if $f(z) = z - \bar{z}$. **[8 + 2]**

OR

Define harmonic function and harmonic conjugate. If a function $f(z) = u(x, y) + i\,v(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.
Show that $u(x, y) = y^3 - 3xy^2$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ of $u(x, y)$. **[2 + 3 + 5]**

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Tribhuvan University

Institute of Science and Technology

Course of Study for Four Year Mathematics

Course Title: Bio Mathematics (Elective) **Full Marks:** 100
Course No. : MAT407 **Pass Mark:** 35
Level : ** B.Sc. **Year:IV
Nature of Course: Theory **Lectures:150 Hrs.**

Course Description

This course is designed for fourth year of Four years B.Sc. program as an elective subject. The main aim of this course is to provide knowledge of Mathematics in Biology.

Course Objectives: The objective of this course is to acquaint students with the basic concepts of mathematics, in population modeling and disease modeling. Also, the idea of solving biological problems.

Course Contents

Unit 1. **Dynamic Modeling with Difference Equations:** The Malthusian Model, Nonlinear Models, Analyzing Nonlinear Models, Variations on the Logistic Model, Comments on Discrete and Continuous Models [20 Lectures]
Unit 2. Linear Models of Structured Populations: Linear Models and Matrix Algebra Projection Matrices for Structured Models, Eigenvectors and Eigenvalues, Computing Eigenvectors and Eigenvalues [20 Lectures]
Unit 3. Nonlinear Models of Interactions: A Simple Predator–Prey Model, Equilibria of Multipopulation Models,Linearization and Stability , Positive and Negative Interactions. [20 Lectures]
Unit 4. Modeling Molecular Evolution: Background on DNA, An Introduction to Probability, Conditional Probabilities, Matrix Models of Base Substitution, Phylogenetic Distances. [20 Lectures]
Unit 5. Constructing Phylogenetic Trees: Phylogenetic Trees, Tree Construction: Distance Methods – Basics, Tree Construction: Distance Methods – NeighborJoining. [20 Lectures]
Unit 6. Genetics: Mendelian Genetics, Probability Distributions in Genetics. [15 Lectures]
Unit 7. Infectious Disease Modeling: Elementary Epidemic Models, Threshold Values and Critical Parameters, Variations on a Theme, Multiple Populations and Differentiated Infectivity. [20 Lectures]
Unit 8. Curve Fitting and Biological Modeling: Fitting Curves to Data, The Method of Least Squares, Polynomial Curve Fitting. [15 Lectures]

Text Books/Reference Books

1. Elizabeth S. Allman, and John A. Rhodes,*Mathematical Models in Biology An Introduction,*Cambridge University Press**,** 2004.

2. Nicholas F. Britton**,** *Essential Mathematical Biology*,Springer-Verlag London Limited 2003.

Unit 1. **Dynamic Modeling with Difference Equations:**

Introduction: Mathematics plays an important role in understanding the biological phenomena. Mathematical models used to describe biological problems are referred as **biomathematics**. In applied mathematics, bio-mathematics is an emerging and dynamics field. There are a large number of biological problems which can enrich mathematics solutions.

Difference Equations

A difference equation is a formula expressing values of some quantity Q in terms of Previous values of Q. Thus, if F(x) is any function, then

$$Q_{t+1} = F(Q_t)$$

is called a difference equation. Difference equation is a recurrence relation.

Example: Let a relation be

$y_n = an + 2$ Where a is any arbitrary constant,

$$y_{n+1} = a(n+1) + 2$$

$y_{n+1} = \frac{y_n - 2}{n}(n+1) + 2$, eliminating a.

Population Growth Model: Discrete Time Data

Let P_t be the population size at time t and let births and deaths in the interval of time $(t.t+1)$ be proportional to P_t. Then, the population P_{t+1} at time $t+1$ is given by

$P_{t+1} = P_t + bP_t - dP_t = (1 + \mathrm{C})P_t$

Where, $C = b - d$.

The general solution of equation is given by

$$P_t = (1 + C)^t P_0$$

Which shows that the population increases or decreases exponentially according as C> 0or C< 0. This is associated with Malthus model.

Note: If $(1 + \mathrm{C}) = k$ is the growth (Malthusian growth) rate of the population, then

$$P_{t+1} = \mathrm{k}P_t$$

is the Maithusian equation in discrete time, whose solution is

$P_t = k^t P_0$.

The population goes on increasing for k >1.

Example: The population of tigers in Chitwan National Park is decreasing at therate of 2 percent per year. If 500 is the initial population of tigers in 2017, find the number of tigers t years later.

Solution: Here,

$$P_t = (1 + C)^t P_0 = (1 - 0.02)^t 500 = (0.98)500$$

For $t = 0, 1, 2, \ldots$

$$P_1 = (0.98)500 = 490$$
$$P_2 = (0.98)^2 500 = 480$$
$$P_3 = (0.98)^3 500 = 471$$

and so on.

Similarly, if 2 percent per year is the increase in the number of tigers, we have

$$P_2 = (1.02)^1 500 = 510$$

and so on.

Exercises: 1 -10, 15.

1.2: Non Linear Model: Derivation of logistic model, cobweb plot.

Exercises: 1 – 6,

1.3: Analyzing the non linear model: Steady state, Equilibrium, stability, nonstability.

Exercises: 1,2 , 3, 6,7, 8, 9.

1.4: Variation on the Logistic Model: Introduction

1.5: Comments on Discrete and Continuous Models: Meanings of discrete and Continuous models.

Unit 2. Linear Models of Structured Populations:

2.1: Linear Models and Matrix Algebra:

Example: Suppose we consider a hypothetical insect with three life stages: egg, larva, and adult. Our insect is such that individuals progress from egg to larva over one time step and from larva to adult over another. Finally, adults lay eggs and die in one more time step. To formalize this, let

E_t= the number of eggs at time *t*,

L_t= the number of larvae at time *t*,

A_t= the number of adults at time *t*.

Suppose we collect data and find that only 4% of the eggs survive to become larvae, only 39% of the larvae make it to adulthood, and adults on average produce 73 eggs each. This can be expressed by the three equations

$E_{t+1}= 73A_t$

$L_{t+1}= 0.04E_t$

$A_{t+1}= 0.39E_t$

Combining all we get

$A_{t+3}= (0.39)(0.04)(73)A_t = 1.1388A_t$

where A_t is the number of adults.

Example. Consider the example above, but suppose that rather than dying, 65% of the adults alive at any time survive for an additional time step. Then the model becomes

$E_{t+1}= 0.73A_t$

$L_{t+1}= 0.04L_t$

$A_{t+1}= 0.39L_t + 0.65A_t$

Example. Suppose we are interested in a forest that is composed of two species of trees, with A_t andB_t denoting the number of each species in the forest in year t. When a tree dies, a new tree grows in its place, but the new tree might be of either species. To be concrete, suppose the species$A$trees are relatively long lived, with only 1% dying in any given year. On the other hand, 5% of the species *B* trees die. Because they are rapid growers, the *B* trees, however, are more likely to succeed in winning a vacant spot left by a dead tree; 75% of all vacant spots go to species *B* trees, and only 25% go to species Atrees. All this can be expressed by

$A_{t+1} = (0.99 + (0.25)(0.01))A_t + (0.25)(0.05)B_t$

$B_{t+1} = (0.75)(0.01)A_t + (0.95 + (0.25)(0.75))B_t$

$\therefore A_{t+1} = 0.9925A_t + 0.1025B_t$

$B_{t+1} = 0.0075A_t + 0.9875B_t$

With $A_0 = 10, B_0 = 900$ we can solve the system and find the equilibrium point.

Exercises: 1-9.

2.2 Projection Matrices for Structured Models: The Leslie model, The Usher model, other structured population models.

Exercises: 1, 3, 4,11.
2.3 Eigenvectors and Eigenvalues:
All worked out examples.
2.4. Computing Eigenvectors and Eigenvalues:
Exercises 1, 3, 4

Unit 3. Nonlinear Models of Interactions:
3.1A Simple Predator–Prey Model: 1, 2 , 9
3.2 Equilibria of Multi population Models: Worked out example to find equilibrium points.
3.3 Linearization and Stability: Worked out example.
Exercises: 2(a), 3(a), 7, 8
3.4 Positive and Negative Interactions. Competition**,**Immune system vs. infective agent, Mutualism.

Unit 4. Modeling Molecular Evolution:
4.1Background on DNA,
4.2An Introduction to Probability
Exercises: All.
4.3Conditional Probabilities:
Exercises:1, 2, 7, 8
4.4Matrix Models of Base Substitution
Markov Models, The Jukes - Cantor Model, Worked out examples
Exercises:1,2
4.5Phylogenetic Distances. Jukes-Cantor distance, The Kimura distances
Exercises:1, 2, and related problems.

Unit 5. Constructing Phylogenetic Trees:
5.1 Phylogenetic Trees
Exercises 1-5.
5.2Tree Construction: Distance Methods – Basics,
Exercises:1, 2, 3
Tree Construction: Distance Methods – NeighborJoining.

Unit 6. Genetics:
6.1 Mendelian Genetics: Dominant and Recessive trait, Mendal's and Data, Punnett Square, Principle of segregation
Exercises:1, 2, 3, 4, 8
6.2Probability Distributions in Genetics.
Exercises: 1-20

Unit 7. Infectious Disease Modeling:
7.1 Elementary Epidemic Models: SIR Model, Difference equation modeling
Exercises:1-6
7.2Threshold Values and Critical Parameters:
Analysis of SIR Model, Basic Reproduction Number (R_0), Interpretation of (R_0) as the average number of secondary infections that would be produced by one infective in a wholly susceptible population of size (R_0), the severity and duration of epidemics.
Exercise: 1-5
7.3 Variations on a Theme: SI and SIS model and their comparison with SIR,

Contact rate and contact number, Immunization strategies
Exercise: **1,2,3,5,7,8**
7.4 Multiple Populations and Differentiated Infectivity:
Exercises:1,2,3,4

Unit 8. Curve Fitting and Biological Modeling:
8.1 Fitting Curves to Data
Exercises:All
8.2 The Method of Least Squares
Exercises: **1-7, 9, 12**
8.3 Polynomial Curve Fitting
Exercises:1-4

Note: There will be 10 questions each carrying 10 marks., from 8 units. From 1 and 2 units, 2 questions are to be made from each unit.All the questions are compulsory. There will be **four Or** choices in any question number from the same unit. The examination period of Math 407 will be 3 hours.

On the basis of the guidelines mentioned, we enclose one set of model question for Bio Mathematics (MAT 407).

LECTURE PROGRAM ON

MATHEMATICAL ECONOMICS (COURSE NO. MAT 408)

FOR

FOUR YEAR B.SC. MATHEMATICS

(IV YEAR)

TRIBHUVAN UNIVERSITY

Prepared by Central Department of Mathematics,
Tribhuvan University, Kirtipur, Kathmandu, Nepal

TRIBHUVAN UNIVERSITY
Institute of Science and Technology
Macro Syllabus
Mathematical Economics

Course Title: Mathematical Economics (Elective) | **Full Marks:** 100
Course No.: Math 408 | **Pass Marks:** 35
Level: B.Sc. | **Year:** IV
Nature of Course: Theory | **Period Per Week:** 9 Hrs

Course Objectives: This course is designed for fourth year of Four years B.Sc. program as an elective subject..This course aims to introduce mathematical modes of economic problems in the society and industry. The students will be able to understand modeling techniques of different economic problems and apply mathematical tools to solve them theoretically. The focus has been given to the equilibrium, cooperative-static and dynamic analysis, and optimization techniques in theory and practice.

Unit 1

Concept of modeling, meanings of mathematical economics and econometrics, equilibrium analysis, market equilibrium linear, non-linear and general models, graphical solutions, solution techniques and applications to national-income analysis and finite Markov chains, non singularity of a matrix and applications to market and income models. [30 Lectures]

Unit 2

Modeling of the marginal and average revenue functions, relationships between the cost functions, geometric interpretations of partial derivatives in economic terms, gradient vector of the production function, applications to comparative static analysis, comparative static analysis of general function models and application to economical problems. [30 Lectures]

Unit 3

Optimization methods for equilibrium analysis, types of extreme points, first, second and nth derivative tests, necessary and sufficient conditions, conditions for profit maximization, the growth function and its variants, extensions to multivariables, conditions to convexity and concavity and economic applications. [30 Lectures]

Unit 4

Effects of constraints in optimization methods, stationary values, Lagrange-multiplier method, second order conditions, multivariable and multi constraints, quasiconcavity and quasiconvexity, utility maximization and consumer demand, changes in price and demands, NLP and KKT-conditions, the constraint qualifications, sufficient conditions to non-linear programming and economic applications. [30 Lectures]

Unit 5

Meaning of economic dynamics, economic applications of integrals, the growth model, dynamics of market price, the qualitative-graphic approach, the market model, Inflation and unemployment models, applications of difference and higher order differential equations to economic problems.

. [30 Lectures]

COURSE DETAILS

Unit 1

1. Concept of modelling: Meaning and importance of mathematics in economics, Mathematical versus non-mathematical economics; Mathematical economics versus econometrics; Ingredients and construction of model.

2. Meaning of equilibrium; Equilibrium models: Linear, non-linear and general (two-commodity and n-commodity).

3. Solution method for linear model: Graphical and elimination of variables.

4. Solution method for non-linear equilibrium model: Graphical, quadratic formula and higher order polynomial technique.

5. Solution method for general equilibrium model: Two-commodity (graphical and elimination method).

6. Application of linear, non-linear and general models in national-income analysis.

7. Test of non-singularity by use of determinant (evaluating an nth- order determinant by Laplace expansion) and rank of matrix and Finite Markov chains.

8. Application of matrix and Markov chains in market and national income models: Market model; National-income model; IS-LM model (closed economy); Leontief input-output model (structure of an input-output model, the open model, a numerical example, the assistance of non-negative solutions, economic meaning of the Hawkins-Simon condition and the closed model); Limitation of static analysis.

9. Related exercises.

Unit 2

1. Concept and modelling of cost and revenue functions; Average cost; Average revenue; Marginal cost and marginal revenue.

2. Relationship between marginal cost and average cost function; Exercise 7.2 (1, 2, 4, 5, 10).

3. Partial differentiation; Geometrical meaning of partial differentiation in economics, Gradient vector, Gradient of production function, Exercise 7.4 (4, 5, 6).

4. Total derivative of implicit function.

5. Application to comparative-static model: Market model, National-income model; Input-output model.

6. Jacobian determinants.

7. Exercise 7.6 (1, 2).

8. Comparative static analysis of general function models and application to economical problems.

9. Market model: Simultaneous-equation and use of total derivatives.

10. National-income model (IS-LM) and extension of the model to open economy.

11. Open economy equilibrium.

12. Exercise 8.6 (1, 2, 3, 4, 5, 6).

13. Limitation of comparative statics.

Unit 3

1. Optimization approach for equilibrium analysis; Optimum and extreme values; Relative minimum and maximum: First derivative test; Relative versus absolute extremum with related examples.

2. Exercise 9.2(4).

3. Relative minimum and maximum: Second and higher order test.

4. Second order test: Necessary versus sufficient conditions; Condition for profit maximization; Example 3.

5. Maclaurin and Taylor series; n^{th} derivative test for relative extremum of a function of one variable.

6. Growth functions: Exponential and logarithm functions.

    (a) Economical interpretation of e.

    (b) Interest compounding and the instantaneous growth.

    (c) Continuous versus discrete growth.

(d) Discounting and negative growth.

(e) Common and natural logarithms.

(f) Base conversion of both exponential and logarithmic functions.

7. Application of growth function

   (a) Optimal timing: A problem of wine storage; Maximization conditions; A problem of timber cutting.

   (b) Finding the rate of growth; Rate of growth of a combination of functions; Finding the point of elasticity.

8. Multi variable optimization

   (a) The differential version of optimization: First-order condition; Second-order condition; Differential conditions versus derivative conditions.
   (b) Extreme values of a function of two variables: First-order condition; Second order partial derivatives; Second-order total differential; Second-order condition; Example 4 and 5.

   (c) Quadratic forms-an excursion: Second-order total differential as a quadratic form; Positive and negative definiteness; Determinantal test for sign definiteness; Three-variable quadratic forms; n-variable quadratic forms; Characteristic-root test for sign definiteness.
   (d) Objective functions with more than two variables: First-order condition for extremum; Second-order condition; n-variable case.
   (e) Second-order conditions in relation to concavity and convexity: Checking concavity and convexity; Differentiable functions; Convex set versus convex function.

9. Economic applications: Problem of multi-product firm; Price discrimination, Input decision of a firm

10. Comparative-static model Aspect of Optimization: Reduced firm solutions; General-function models.

11. Exercise 11.7 (1, 2, 4, 5).

Unit 4

1. Optimization with equality constraints: Effect of constraint; Finding the stationary values (Lagrange-multiplier method; Total-differential approach; An interpretation of Lagrange multiplier; n-variable and multi-constraint cases).

2. Exercise 12.2(2, 3, 6).

3. Second-order conditions: Second order total differential; Second order conditions; The bordered-Hessian; n-variable case, multi-constraint case.

4. Exercise 12.3(1, 4).

5. Quasiconcavity and quasiconvexity: Geometrical characterization; Algebraic definition; Theorem I, II, III; Differential functions; Absolute versus relative extrema.

6. Quasiconcavity and quasiconvexity of differential functions with examples.

7. Exercise 12.4 (1, 2, 8).

8. Utility maximization and consumer demand: First order condition; Second order condition; Comparative-static analysis; Proportionate change in price and income.

9. Cobb-Douglas production function; Extensions of the results.

10. Least-cost combination: First-order condition; Second-order condition; The expansion path; Homothetic functions; Elasticity; CES production function; Cobb-Douglas function as a special case of the CES function.

11. Non-linear programming and KKT conditions: Effect of non-negativity restrictions; Effect of inequality constraints; Interpretation of the Khun-Tucker conditions; The n-variable, m-constraints case.

12. Constraint qualification: Irregularities at boundary points constraint qualification; The constraint qualification; Linear constraints.

13. Economic application: War-time rationing; Peak-load pricing.

14. Sufficiency theorem in NLP: The Kuhn-Trcker sufficiency theorem: concave programming; The Arrow-Enthoven sufficiency theorem: Quasiconcave programming; A constraint qualification test.

Unit 5

1. Economic Dynamics and integral, meaning of economic dynamics.

2. Economic application of integral: From a marginal function to total function; Investment and capital formation; Present value of cash flow, present value of a prepetual flow.

3. Domor growth model: The framework; Finding the solution; The razor's edge.

4. Dynamics of market price: The framework; The time path; The dynamic stability of equilibrium; An alternative use of the model and its solutions.

5. The qualitative-graphic approach: the phase diagram, types of time path.

6. Market model with price expectations: Price trend and price expectations; A simplified model; The time path of price; Example 1, 2.

7. The interaction of inflation and unemployment: The Phillips relation, the expectation-augmented Phillips relation; The feedback from inflation to unemployment, the time path of π, Example 1.

8. Discrete time, difference, difference equations and higher order differential and difference equation and their applications.

9. The dynamic stability of equilibrium: The significance of b; Convergence to equilibrium.
10. The cobweb model: The model; The cobwebs.
11. A market model with inventory: The model; The time path; Graphical summary of the results.
12. Non-linear Difference equations-The qualitative-graphic approach: Phase diagram; Types of time path; A market with a price ceiling.
13. Samuelson multiplier-acceleration interaction model: The framework; The solution; Convergence versus divergence; A graphical summary.
14. Inflation and unemployment in discrete time: The model; The difference equation in p; The time path of p; The analysis of U; The long-run Phillips relation.
15. Dynamic input-output models: Time lag in production; Excess demand and output adjustment; Capital formation.

Text/Reference Book(s):

1. A. C. Chiang, Kervin Wainwright; Fundamental Methods of Mathematical Economics, McGraw Hill Publishers.

2. Hoy Michael, et al.; Mathematics for Economics, Third edition, PHI Learning Private Limited.

Tribhuvan University

Institute of Science and Technology

Course Title: Mathematical Modeling	Full Marks: 50
Course No: MAT 409	Pass Marks: 17.5
Level: B. Sc	Year: IV
Nature of course: Theory	Period per Week 4Hrs

Unit 1: Modeling Change: 15 Lectures

Introduction, Mathematical models, Modeling change with difference equations, Approximating change with difference equations, Solution to dynamical systems, Systems of difference equations.

Unit 2:The Modeling Process, Proportionality and Geometric Similarity: 15Lectures

Mathematical models, modeling using proportionality, modeling using geometric Similarity, Automobile gasoline mileage, Body weight and height, Strength andagility.

Unit 3: Model Fitting: 15 Lectures

Fitting model to data graphically, Analytical methods of data fitting, Applying the least squares criterion, Choosing a best model.

Unit 4: Optimization of Discrete Models: 15 Lectures

Introduction, An overview of optimization modeling, Linear programming, Geometric, Algebraic, Simplex method.

Unit 5: 15 Lectures

Population growth, Prescribing drug dosage, Braking distance revisited. Graphical Solutions of autonomous Differential equations.

Introduction

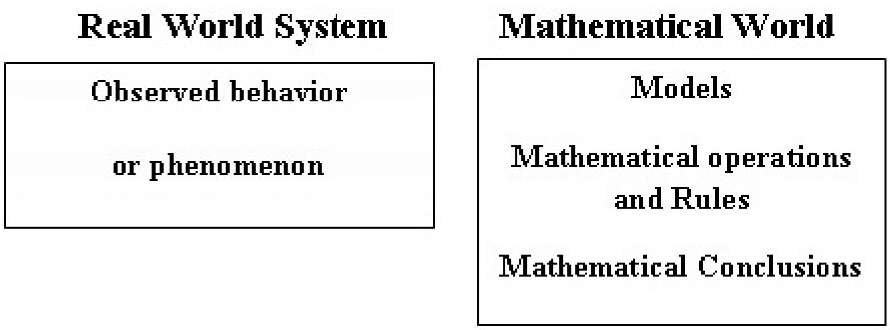
What is Modeling? Modeling of devices and phenomena is essential to both engineering and science.So engineers and scientists have very practical reasons for doing mathematical modeling.

Real World: At the beginning, one has to identify **a real world (or external world)**
Conceptual World: a conceptual world (or mathematical world)

Introduction

- **Real World:**
  Various phenomena and behaviors, whether natural in origin or produced by artifacts,are observed.

- **Conceptual World:**
  The world of mind, which deals with the observation, modeling and prediction of the phenomena and behaviors which are happening in the real world.



Principles of Mathematical Modeling

1. Problem Identification
2. Making Assumptions; Variables Selection and their interrelationship
3. Solving and interpretation of the model
4. Verification of the model
5. Implementation of the model
6. Maintaining the model.

4, 5 and 6 steps are important for the stability of the solution of the modeled problem.

Where and when Mathematics can be used?

- There is no discipline free from mathematics. So we claim that mathematics is everywhere.

- Mathematics used to solve physical problems is called an applied mathematics.
- A physical problem means the problem in our surroundings.
  A stone thrown into the air,
  birds flying in the sky and fishes swimming in the water
  can be described by mathematics.

Where is Mathematics?

- Inside our room, we find cuboids beams, rectangular windows.
- The corners of the room can be considered as the origin and the surfaces are perpendicularto each other at the origin. One of the corners can be considered as thex- axis and other as the y-axis.
- We see the cracks in the field when there is no rain. The cracks in the ground areof different shapes which can be rectangular, cuboids, cubic, hexagonal or otherirregular shapes.
  Also the trees on the sides of the road.

Example

- *Using trigonometry, the height of the tree can be approximated. This idea is common in forest officers*.
- *The volume and surface area of a human can be estimated crudely but quickly usingthe formula for the cylinder.*

For example;
If $r = 1/2$ft. and $h = 5$ft. then, $s = 2\pi rh$ and $v = \pi r^2 h$ give $s \approx 16 ft^2$ and $v \approx 4 ft^3$

Main Steps for Modeling

The method of describing the problems is called modeling. For the modeling we need to follow the following steps;

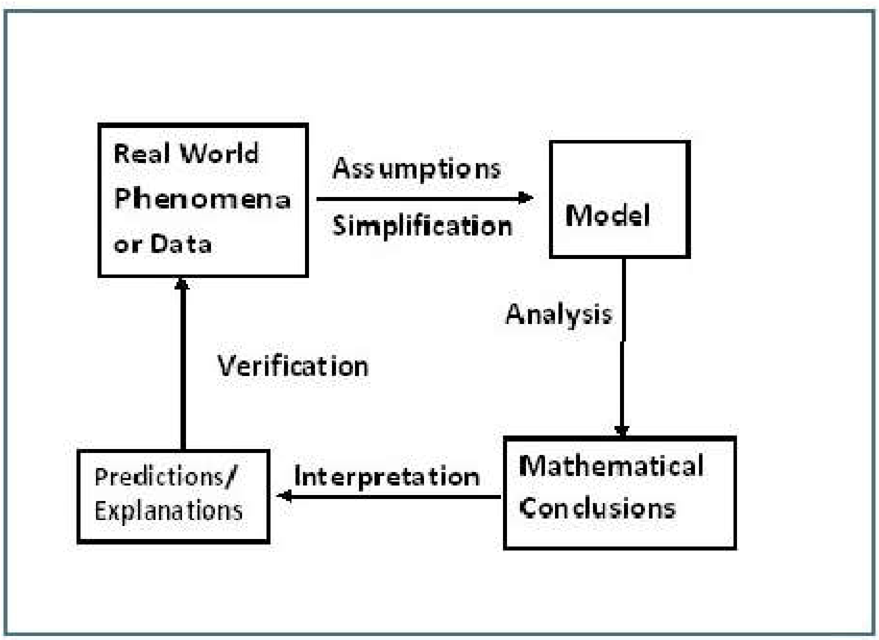
1. Problem and set up a model
2. Interpretation of the problem in real terms
3. Stability validity of the solution.

Examples of Modeling
Example 1: The product of the age of a father and a son is 800. Find their ages if the son
is 20 years younger than the father.
Solution: Setting a model: Let x is the age of the son then $x(x + 20) = 800$
Solving we get
$x = 20$ and $= -40$: Analysis of the problem.
$x \neq -40$,
$\therefore x = 20$: Interpretation and validations of the problem
Use of Mathematics in Biology

- **External Bio-fluid Dynamics:** People invented big airplanes seeing the birds flyingin the sky.

Example: Eagles and other big birds
Small planes like helicopters were invented: seeing the small birds like humming birds.
Seeing the fish in the water: ships and marines were invented.

- **Internal Bio-fluid Dynamics:** Flow of blood, urine, cerebro-spinal and transport of mass and heat within an animal and flow of air in lungs.



Detailed Course:

Unit 1 Modeling Change

In modeling our world, we are often interested in predicting the value of a variable at sometime in the future. Examples are:

- population after some years
- real state value after some years
- the number of people with a communication disease after some time.
-

That is, mathematical model can help us to understand a behavior better or aid us in planningfor the future. See the previous figure.

Simplification:

Models can only approximate real world behavior. One very powerful simplifying relationshipis proportionality:

Definition: Two variables y and x are proportional (to each other) if one is always a constantmultiple of the other- that is, if

$$y = kx$$

for some non zero constant k. We write$y\ \alpha x$

1.1 Modeling Change with Difference Equation
Introduction to Difference Equation

A powerful paradigm to use in modeling change is

future value = present value + change

By collecting the data over a period of time and plotting those data, we may capture the trend of the change in the behavior. If the behavior is talking place over discrete pathperiods then we are concerned with **difference equation**. If the behavior is taking placecontinuously with respect to time then we are concerned with the **differential equation.**

Definition: For a sequence of numbers $A = \{a_0\ ,\ a_1, a_2 \dots\}$the first differences are

$$\Delta a_0 \ = a_1 - \ a_0$$
$$\Delta a_1 \ = a_2 - \ a_1$$
$$\Delta a_2 \ = a_3 - \ a_2$$
$$\dots$$
$$\Delta a_n \ = a_{n+1} - \ a_n$$

na positive integer.

Example: A Saving Certificate Consider the value of a savingscertificate initially worth *Rs*1000 that accumulates interest paid each month at 1 percentper month. Then

$$A \ = \ \{1000, 1010, 1020.10, 1030.30, \ \ . \ . \ .\}$$

The first differences are

$$\Delta a_0 \ = a_1 - \ a_0 \ = 1010 - 1000 = 10$$
$$\Delta a_1 \ = a_2 - \ a_1 = 1020.20 - 1010 = 10.10$$
$$\Delta a_2 \ = a_3 - \ a_2 = 1030{,}30 \ - 1020.10 = 10.20$$

Example: A Saving Certificate (continuous)

Note that the first differences represent the damage in the sequence during one time period,or the interest.The first difference is useful for modeling change talking place in discrete intervals. Here,the change in the value of the certificate from one month to the next month is merely theinterest paid during that month. If n is the number of months and a_n the value of thecertificate after n months, then the change or interest growth in each month is representedby the *nth* difference

$$\Delta a_n \ = a_{n+1} - \ a_n = 0.01a_n$$

which can be rewritten as

$$a_{n+1} = \ a_n + 0.01a_n$$

Example: A Saving Certificate (continuous)

As we know that the initial deposit is Rs1000 we have the dynamical system model

$$a_{n+1} = 1.01a_n \,, \qquad n\ 0, 1, 2, \ \ . \ . \ .$$
$$a_0 = 100$$

APPROXIMATING CHANGE WITH DIFFERENCE EQUATIONS

If *Rs*50 is drawn from the account each month, the change during a period would be theinterest earned during that period minus the monthly withdrawal. That is,

$$\Delta a_n \ = a_{n+1} - \ a_n = 0.01a_n - 50$$

Example: Mortagaging a Home , See the book.

Definitions

A **sequence** is a function whose domain is the set of all nonnegative integers and whoserange is a subset of the real numbers.

A **dynamical system** is a relationship among terms in a sequence.

A **numerical solution** is a table of values satisfying the dynamical systems.

1.2 Approximating Change with Difference Equations

In most of the examples, describing the change mathematically is not precise. For suchconditions, we must plot the change, observe a pattern, and then approximate the changein mathematical terms. Here we approximate some observed change to complete theexpression

$$Change = \ \Delta a_n = some\ function\ of\ f$$

Example: Growth of a Yeast Culture:

Data collected in a yeast culture experiment is given in the table. The graph representsthe assumption that the change in population is proportional to the current size of thepopulation. That is,

$$\Delta P_n \ = P_{n+1} - \ P_n = kP_n$$

whereP_n represents the size of the population biomass after n hours, and k is a positiveconstant. The value of k depends on the time measurement.

Example 1: Growth of a Yeast Culture:

Model Refinement: ModelingBirths, Deaths, and Resources:

If both births and deaths during a period are proportional to the population, then thechange in population should be proportional to the population, as in the Example 1. However, certain resources (e.g., food) can support only a maximum population level ratherthan one that increases indefinitely. As these maximum levels are approached, growthshould slow. These are described in the graphs.

Example 2: Growth of Yeast Culture Revisited

The data in the figure shows change in biomass beyond the eight observations. Here, inthe third column of the data note that the change population per hour becomes smaller asthe resources become more limited or constrained. We see the population appears to beapproaching a limiting value, or **carrying capacity**, may guessed from figure as 665. As P_n approaches 665 the change is slow. Because 665 -P_n gets smaller as P_n approaches665, we propose the model

$$\Delta P_n = P_{n+1} - P_n = k(665 - P_n)P_n$$

Example 2: Growth of Yeast Culture Revisited

The value of *k* can be estimated by 0.00082, giving the model

$$\Delta P_n = P_{n+1} - P_n = 0.00082(665 - P_n)P_n$$

Solving the model numerically for P_{n+1}, we have

$$P_{n+1} = P_n + 0.00082(665 - P_n)P_n$$

Model Refinement: Modeling Births, Deaths, and Resources:

Yeast biomass approaching a limiting population level

1.3 Solution to Dynamical Systems

Example:Saving Certificate Revisited

In the saving certificate example a savings certificate initially worth Rs1000 accumulatedinterest paid each month at 1percent of the balance. No deposit or withdrawals occurredin the account, determining the dynamical system

$$a_{n+1} = 1.01\, a_n\,,\ a_0 = 1000 \qquad (1.1)$$

From the graph we see that the sequence$\{a_0, a_1, a_2, \ldots\}$ grows without bound.

Example:

Saving Certificate Revisited

Algebraically, we see the graph pattern

$$a_1 = 1010.00 = (1.01)a_0 = (1.01)1000$$
$$a_2 = 1020.10 = (1.01)a_1 = (1.01)^2 1000$$
$$a_3 = 1030.10 = (1.01)a_2 = (1.01)^3 1000$$
$$a_4 = 1040.10 = (1.01)a_3 = (1.01)^4 1000$$

The pattern of the sequence suggests that thekthterma_k is the amount 1000 multiplied by $(1.01)^k$.

Example:Saving Certificate Revisited

Conjecture: For $k = 1, 2, 3,\ldots$ the term a_k in the dynamical system (1.1) is

$$a_k = (1.01)^k 1000 \qquad (1.2)$$

Test the Conjecture: We test the conjecture by examining whether the formula for $a_k$satisfies the system (1.1) upon substitution.

$$a_{n+1} = (1.01)a_n$$
$$(1.01)^{n+1}1000 = (1.01)[(1.01)^n 1000] = (1.01)^{n+1}1000$$

Since the last equation is true for every positive integer n, the conjecture is accepted.

Example:Saving Certificate Revisited

Conclusion: The solution for the term a_k in the dynamical system (1.1) is

$$a_k = (1.01)^k 1000$$
$$\text{or } a_k = (1.01)^k a_0$$

which computes the balance a_k in the account after k months.

Linear Dynamical Systems$\boldsymbol{a_{n+1} = r a_n}$**, for** r **Constant**(see the book)

Example 2: Sewage Treatment Long Term Behavior of$\boldsymbol{a_{n+1} = r a_n}$, for *r* Constant

Example 3: Prescription for Digoxin, Example 4: An Investment Annuity, Finding andClassifying Equilibrium Values

Theorem 2, 3, Example 6

1.4 System of Difference Equations

Example 1: Car Rental Company, Example 2: The Battle of Trafalgar **Exercises on Unit 1: 1.1:** 1,2,3, 4,5. **1.2:** 1, **1.3:** 1, 2, 3, **1.3:** 1

Unit 2:The Modeling Process, Proportionality and Geometric Similarity

2.1 Mathematical Models

Construction of Models: The following steps are important while constructing a model:
1. Identify the problem.
2. Make assumptions
(a)Identify and classify the variable. (b)Determine interrelationships between the variables and submodels
3. Solve the model
4. Verify the model
(a)Does it address the problem? (b)Does it make common sense? (c) Test it with real world data.
5. Implement the model.
6. Maintain the model.

2.2 Modeling Using Proportionality

Modeling Using Proportionality: Examples from the book.Modeling Vehicular Stopping Distance.

2.3 Modeling Using Geometric Similarity

Modeling Using Geometric Similarity: Definition, Example 1: Raindrops from a Motionlesscloud. Testing Geometric Similarity.
Example 2: Modeling a Bass Fishing Derby.

2.4 Automobile Gasoline Mileage

2.5 BodyWeight and Height, Strength and Agility

Unit 3 Model Fitting

3.0.1 Introduction

Introduction, Relationship Between Model Fitting and Interpolation,Sources of Error inthe Modeling Process.

3.0.2 Fitting Models to Data Graphically

Visual Model Fitting with the original Data, Transforming the Data

3.1 Analytical Method of Model Fitting: Related problems

Chebyshev Approximation Criterion, Least Square Criterion, Relating the Criteria

3.2 Applying the Least Square Criterion

Fitting a Straight Line, Fitting a power Curve
Related problems.

Unit 4 Optimization of Discrete Models

Introduction, Overview of Optimization Modeling. Example 1
Classifying Some Optimization Problems, Unconstrained Optimization Problems, Integer
Optimization Programs: Examples 2, 3
Multiobjective Programming: An Investment Problem, Dynamic Programming Problems

4.1 Linear Programming 1: Geometric Solutions
Interpreting a Linear Program Geometrically
Example 1: Carpenter's Problem, Example 2: Data Fitting Problem, Model Interpretation,
Empty and Unbounded Feasible Regions, Level Curves of the Objective Function.
Theorem 1 (No Proof)

4.2 Linear Programming II: Algebraic Solutions
Linear Programming II: Algebraic Solutions. Example 1 Solving the Carpenter's Problem Algebraically

4.3 Linear Programming III: The Simplex Method
Linear Programming III: The Simplex Method, Example 1 Carpenter's Problem Revisited.
Example 2 and Related Examples

Unit 5 Modeling using Differential Equations

Introduction: The Derivative as a Rate of Change, The Derivative as the Slope of the TangentLine, verifying the Model, Refining the Model to Reflect Limited Growth, Verifying the Limited Growth Model.

5.1 Population Growth
Malthus model: The equation $\dot{x}$ = ax, a > 0 represents is a simplest model for thepopulation growth, where x(t) measures the population of some species at time t. Theequation tells that the growth rate of population is directly proportional to the size ofthe population. Such model does not consider the different circumstances like, famine,diseases, war etc, which are the bounds in the increment of the population. To describethe circumstances, Logistic model is used.

Example 5.1.1.The Logistic Population Model
The model considers the followings:

1. If the population is small, the growth rate is nearly directly proportional to the sizeof the population.

2. If the population is too large, the growth rate became negative.
The differential equation satisfying the assumptions is
$\dot{x} = ax(1 - x/N)$ *(5.1)*
Where$a > 0$ is a parameter, gives the rate of population growth when x is small, while$N > 0$is a parameter, represents carrying capacity of the population.If x is small,
$$\dot{x} = ax$$

If$x > N$, $\dot{x}$< 0. Without loss of generality, let $N = 1$, then
$\dot{x} = ax(1 - x)$(5.2)
is a first order, autonomous, nonlinear equation. The solution of this equation is
$$x(t) = \frac{Ke^{at}}{1 + Ke^{at}}$$

where K is determined at time t = 0 as

$$K = \frac{x(0)}{1 - x(0)}$$

$$x(t) = \frac{x(0)e^{at}}{1 - x(0) + x(0)e^{at}}$$

Exercises: 1, 2, 3, 4, 6, 7, 8.

5.2 Prescribing Drug Dosage
Prescribing Drug Dosage
Exercises: 1, 2, 3, 4

5.3 Braking Distance Revisited
Braking Distance Revisited
Worked out example

5.4 Graphical Solution of Autonomous Differentia Equations
Equilibrium values or rest points, Examples 1, 2, 3
Exercises on 11.4: 1-9.
Further reading: Example 9, 10

Text/Reference Books:

1. Frank R. Giordano, William P. Fox, Steven B. Horton, Maurice D. Weir; *Mathematical Modeling, Principles and Applications*, Cengage Learning, India Edition.
2. Sandip Banerjee; *Mathematical Modeling, Models, Analysis and Applications*, CRC press. A Chapman and Hall Book, 2014.
3. Boyce, W. and DiPrima, R.; *Elementary Differential Equations and Boundary Value Problems*, 9th Ed., Wiley India.

Guidelines to the question setter

There will be 5 questions each carrying 10 marks. All the questions are compulsory. There will be **two** OR choices in any question number from the same unit. The examination period of Math 407 will be 2 hours.

On the basis of the guidelines mentioned, we enclose one set of model question for Mathematical Modelling (Math 407)

MODEL QUESTION

Tribhuvan University

Bachelor Level / IV year/ Sc. & Tech. Full Marks: 50

Mathematical Modeling (MAT 409) Time: 2 Hours

Candidates are required to give their answers in their own words as far as practicable. Attempt ALL the questions.

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1. (a) Write out the first five terms a_0- a_4 of $a_{n+1} = a^2_n$, $a_0 = 1$ [3]

(b) Write out the first four algebraic equations of $a_{n+1} = 3a_n, a_0 = 1$ for n =0, 1, 2, 3. [3]

(c) If you currently have Rs5000 in a saving account that pays 0.5 percent each month and you add Rs200 each month, formulate a dynamical system. [4]

OR

(a) For the linear dynamical systems $a_{n+1} = ra_n$, for r constant, prove that $a_k = r^k a_0$, where a_0 is a given initial value. [5]

(b) A sewage treatment plant processes raw sewage to produce usable fertilizerand clean the water by removing all other contaminants. The process is suchthat each hour 12 percent of remaining contaminants in a processing tank areremoved. What percentage of the sewage would remain after 1 day?How long would it take to lower the amount of sewage by half? How longuntil the level of sewage is down to 10 of the original level? [5]

2. (a) When two objects are geometrically similar? If two cuboids are considered,whose length breadth and height are l, b, h and l', b', h'. Prove that if y =f(l, S, V), where l, S, V represent lenght, surface area and volume , respectively,then $y = g(l, l^2, l^3)$ [5]

(b) Find the terminal velocity of a rain drop from a motionless cloud, assuming theconstant gravity. [5]

3. Classify the describe the different types of errors. Fit $y = Ax^2$ to the data and predict the value at x = 2:25

x:0.5 1.0 1.5 2.0 2.5

y:0.7 3.4 7.2 12.4 20.1 [5 + 5]

4. Solve the linear program represented bythe data given and the model y = c*x* with the largest absolute deviation ri = |yi- y(xi)|

x:1 2 3

y:2 5 8 [10]

5. (a) Give mathematical model for the population growth by Malthus. Explain why this model is not realistic. [5]

(b) Consider the model for the cooling of a hot cup of soup;

$$\frac{dT_m}{dt} = -k(Tm - \beta),\ k > 0$$

where Tm(0) = α. Here Tm is the temperature of the soup at any time t >0, β is the constant temperature of the surrounding medium, α is the initialtemperature of the soup, k is a constant of proportionality depending on thethermal properties of the soup. Find Tm. [5]

OR

Apply the phase line techniques to obtain solution curves for the logistic growth equation $\frac{dP}{dx} = r(M - P)$ [10]