Model Question

Tribhuvan University

Bachelor Level (4 Yrs.) Sci. & Tech. / III year **Full Mark: 75**
Real Analysis (MAT 202) **Time: 3hrs.**

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt ALL questions.

Group "A" **[5×8=40]**

1. Define bijective function with an example. Is the composite of two functions necessarily Commutative? Prove that if $f: X \to Y$ and $g: Y \to Z$ are bijective, then the composite function $g \circ f: X \to Z$ is also bijective and $(g \circ f)^{-1} = f^{1} \circ g^{-1}$. [1+2+5]

2. Define the interior point of a set in $\mathbb{R}$. Find the interior points of the set $\{1,2,3,4,5,6\}$. Prove that the intersection of finite number of open sets is an open set. Is the intersection of infinite number of open sets is open? If not, justify your answer by a suitable example. [1+2+3+2]

OR

Define adherent point and the closure of a set. Determine the adherent points of the set $\{0\} \cup (1, 7)$. Prove that a set is closed if and only if it contains all its adherent points. [1+2+5]

3. Define convergence of a series. Prove that an infinite series Σx_n converges if and only if for each $\varepsilon > 0$ there is a positive integer N such that

$$|x_{n+1} + \ldots.. + x_m| < \varepsilon \text{ whenever } m > n \geq N.$$

Apply this criterion to determine whether the following series are convergent or divergent.

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots \ldots$ **(ii)** $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots$ [1+3+2+2]

OR

If Σu_n is a positive terms series such that $\lim_{n\to\infty} (u_n)^{1/n} = \ell$, prove that

(i) The series converges if $\ell < 1$
(ii) The series diverges if $\ell > 1$
(iii) No definite information if $\ell = 1$.

Apply this test to examine the convergence or divergence of the series

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots$$ for different values of x. [5+3]

4. Let $f: [a, b] \to \mathbb{R}$ be continuous function and k be a real number between $f(a)$ and $f(b)$. Prove that there exists a point $c \in [a, b]$ such that $f(c) = k$. Give its geometrical meaning.

Use Intermediate Value Theorem to show that $f(x) = x^3 - x$ has a root in the interval [-2, 2]. [3+2+3]

5. Define upper and lower Riemann sums of a bounded function. Prove that a necessary and sufficient condition for a bounded function $f: [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$ is that for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

   $0 \le U(P, f) - L(P, f) < \varepsilon.$ [3+5]

Group "B" [5×7 = 35].

6. Define supremum and infimum of a set of real numbers $\mathbb{R}$ with an example of each. State the completeness axiom of the real number system. Prove that if a and b are real numbers such that $a < b$, there is a rational number r such that $a < r < b$. [2+1+4]

OR

Define countable and uncountable sets with an example of each. Prove that the set of all real numbers between 0 and 1 is uncountable. [2+5]

7. Define monotonic sequence in $\mathbb{R}$ with an example. If the sequence is monotonically increasing and bounded above, then prove that it is convergent and attains its least upper bound.

   Test whether the sequence $\left\{\frac{n+1}{n^2+2}\right\}$ is monotonic or not. [2+3+2]

8. Let f, g and h be functions defined on some open interval (a, b) containing a point c, and let $f(x) \le g(x) \le h(x)$ except possibly at c.

   If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$, then prove that $\lim_{x \to c} g(x)$ exists and is equal to L. Use this theorem to prove that $\lim_{x \to 0} (x^2 \sin x) = 0$. [5+2]

9. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a number c in (a, b) such that $f'(c) = 0$. Interpret it geometrically. Is the continuity of f at the end-points necessary? Justify your answer. [4+1+2]

10. Define primitives and integrals of a function. Distinguish between them by taking a suitable example. Let f be a function integrable on a closed interval $[a, b]$ and let

    $F(x) = \int_a^x f(u)\, du$, $x \in [a, b]$, prove that $F(x)$ is continuous on $[a, b]$. [1.5 + 1.5 + 4]

OR

State the first and second fundamental theorem of integral calculus. Prove the second fundamental theorem. Evaluate: $\int_0^3 \sqrt{x+1}\, dx$, by using the second fundamental theorem.

[1.5 + 4 + 1.5]

✸